Relativistic Constraint Dynamics and QCD Bound States

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Formulation of relativistic two-body bound-state wave equations and their relationship to Quantum Field Theory still has no generally agreed-upon solution. The appropriate tool to achieve this goal is the Bethe-Salpeter formalism, but its inherent complexity leads to series of difficulties mostly related to the central role played in it by the relative time or energy. We consider bound states in the spirit of "Constraint Relativistic Quantum Mechanics". Relativistic bound-state problem is formulated with the use of symmetries, energy-momentum conservation laws in Minkowskiy space. Interpolating complex-mass formula for two exact asymptotic eigenmass expressions is obtained. To verify the model, relativistic two-body wave equation with position dependent particle masses is used to describe the flavored Qq systems. Solution of the equation for the system in the form of a standing wave is given. Mass spectra for some leading-state flavored mesons are calculated.

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INTRODUCTION

The strict description of bound states in a way fully consistent with all requirements imposed by special relativity and quantum mechanics (QM) is one of the great challenges in theoretical elementary particle physics. Such description can be done within the framework of Quantum Field Theory (QFT). The appropriate tool to achieve this goal is the Bethe-Salpeter (BS) formalism [1]. The manifestly covariant BS equation obtained directly from QFT governs all the bound states.

However, attempts to apply the BS formalism to relativistic bound-state problems lead to a number of various difficulties. These are the impossibility to determine the BS interaction kernel beyond the tight limits of perturbation theory, appearance of abnormal solutions that are difficult to interpret in the framework of quantum physics. The principal difficulty in the treatment of the BS equation comes from the existence of unphysical relative time variables. For various practical reasons, in applications to both quantum electrodynamics (QED) and quantum chromodynamics (QCD), as well as nonrelativistic (NR) reduction, some simplified equations are usually used. Such equations are usually obtained using certain restrictions and constraints.

The usual practice consists in eliminating the relative time variable (3D reduction). The 3D reduction of the two-fermion BS equation has been performed in many works [2–7]; all these methods are theoretically equivalent; there were suggested noncovariant instantaneous truncations of the BS equation [8, 9]. The most well-known of the relativistic bound-state equations is the one proposed by Salpeter [2]. The equivalence of simplified equations with the original BS equation can be proved exactly [10].

The two-fermion relativistic wave equations were considered in the framework of Constraint Theory [11]. The use of the manifestly covariant formalism with constraints in the relativistic bound-state problem [6, 12] leads to a Poincaré invariant description of the dynamics of the system with the correct number of degrees of freedom. The potentials that appear in the corresponding wave equations are calculable in terms of the kernel of the BS equation, and therefore allow one to deal with QFT problems.

In QED, the Coulomb gauge is the most convenient for treating the relativistic bound-state problem, since it allows the optimal expansion of the BS equation around the NR theory [13–15]. The main disadvantage of the Coulomb gauge is, however, its noncovariant nature. Constraint theory, leads to a manifestly covariant 3D description of relativistic two-body (R2B) systems [5, 11, 12, 16–18] and has opened a new perspective.

It was shown [5, 7] that the expansion of the BS equation around the constraint theory wave equations in the Feynman gauge (as well as for scalar interactions) is free of the above mentioned diseases of covariant gauges and allows a systematic study of infrared leading effects of multiphoton exchange diagrams; the latter can then be represented in three-dimensional x-space as local potentials. Summing the series of these leading terms one obtains a local potential in compact form [7], which is well suited for a continuation to the *strong coupling domain* (QCD) of the theory or for a generalization to other effective interactions.

In this work we consider bound states as R2B systems in the potential approach without using the BS equation or its reductions. Difficulties encountered here relate to 1) two-particle relativistic equation of motion and 2) absence of a strict definition of the potential in relativistic theory. We begin our consideration of the R2B problem with relativistic classical mechanics. Relativistic bound state problem is formulated with the use of symmetries, energy-momentum conservation laws in Minkowskiy space. The potential of interaction is treated as the Lorentz-scalar function of the spatial variable r, the distance between particles. The concept of position dependent particle mass is developed. Using the correspondence principle, we deduce, from the R2B classic equation, the two-particle wave equation. The free particle hypothesis for the bound state is developed: particles inside the system move as free ones. Complex eigenmasses for the bound system are obtained. The relative motion of quarks in eigen states is described (in the physical region) by the standing wave of the form $C_n \sin(k_n x + \delta_n)$ for each spatial degree of freedom. To verify the model, the R2B wave equation with position dependent quark masses is used to describe the flavored Qq mesons.

I. BOUND STATES IN CONSTRAINT THEORY

The relativistic bound-state of two scalar particles can be described by two independent wave equations, which are generalizations of the individual Klein-Gordon equation of each particle, including the mutual interaction potential. The compatibility condition of the two equations imposes certain restrictions on the structure of the potential and leads, in a covariant form, to an elimination of the relative energy variable. This results in the manifestly covariant, 3D eigenvalue equation that describes the relative motion of the two particles [7]. This equation is very similar in form to the Schrödinger (or Klein-Gordon) equation: it is a second order differential equation in the three spacelike coordinates and therefore the usual techniques of NR quantum mechanics are applicable to it.

For two fermions, the system is described by two independent Dirac type equations. In this case, the compatibility condition imposes restrictions on the stucture of the potentials and eliminates the relative energy variable; however, because of the Dirac matrices, the reduction to a final eigenvalue equation is not straightforward. The reduction process is rather complicate and depends on the way of eliminating the components of the spinor wave function in terms of one of them. Up to now, no single Pauli-Schrödinger type equation was obtained from this procedure.

Two body BS equation [8, 19, 20] for spin-zero bound states is

$$G_0^{-1}\Psi \equiv (p_1^2 + m_1^2)(p_2^2 + m_2^2)\Psi = K\Psi, \tag{1}$$

where $G_0 = G_{0,1}G_{0,2}$ is free propagator of particles. The irreducible BS kernel K would in general contain charge renormalization, vacuum polarization graphs and could contain s elf-energy terms transferred from the inverse propogators. The kernel K is obtained from the off-mass-shell scattering amplitude,

$$T = K + KG_0T. \tag{2}$$

Recent work with static models has indicated, that abnormal solutions disappear if one includes all ladder and cross ladder diagrams [7]. This supports Wick` s conjecture on defects of ladder approximations. In the mean time numerous 3D quasipotential reductions of the BS equation had been proposed.

Reductions of the BS equation can be obtained from iterating this equation around a 3D Lorentz invariant hypersurface in relative momentum (p) space. This leads to invariant 3D wave equations for relative motion. The resultant 3D wave equation is not unique, but depends on the nature of the 3D hypersurface. One can choose Todorov` s quasipotential equation [5] which has this Schrödinger-like form

$$[\hat{p}^2 + \Phi(x_1 - x_2)]\psi = \kappa^2(w)\psi, \tag{3}$$

where the quasipotential Φ is related to the scattering amplitude, 3D hyperfine restriction on the relative momentum p is defined by $p \cdot P\psi = 0$, $P = p_1 + p_2$. The effective eigenvalue in (3) is

$$\kappa^2(w) = \frac{1}{4w^2}[w^2 - (m_1 - m_2)^2][w^2 - (m_1 + m_2)]^2, \tag{4}$$

with $w = \sqrt{P^2}$ the c.m. invariant energy.

The forces Φ to depend on $x_1 - x_2$ only through the transverse component, $x^\mu_\perp$. Thus, in the c.m. frame, the hypersurface restriction $p \cdot P\psi = 0$ not only eliminates the relative energy $[p\psi = (0, \mathbf{p})\psi = 0]$ but implies that the relative time does not appear $[x^\mu_\perp = (0, \mathbf{r})]$.

II. THE SPINLESS SALPETER EQUATION

Valuable are methods which provide either exact or approximate analytic solutions for various forms of differential equations. They may be remedied in three-dimensional reductions of the BS equation. In most cases the analytic solution can be found if original equation is reduced to the Schrödinger-type wave equation. The most well-known of the resulting bound-state equations is the one proposed by Salpeter [2]. There exist many other approaches to bound-state problem. One of the promising among them is the Regge method in hadron physics [21].

All hadrons and their resonances in this approach are associated with Regge poles which move in the complex angular momentum J plane. Moving poles are described by the Regge trajectories, $\alpha(s)$, which are the functions of the invariant squared mass $s = W^2$ (Mandelstam's variable), where $W = E^*$ is the c. m. rest energy (invariant mass of two-particle system). Hadrons and resonances populate their Regge trajectories which contain all the dynamics of hadron interaction in bound state and scattering regions.

Light and heavy mesons have been studied in a soft-wall holographic approach AdS/CFT [22] using the correspondence of string theory in Anti-de Sitter space and conformal field theory in physical space-time. It is analogous to the Schrödinger theory for atomic physics and provides a precise mapping of the string modes $\Phi(z)$ in the AdS fifth dimension z to the hadron light-front wave functions in physical space-time.

Let us consider and analize general coordinate-space relativistic spinless Salpeter (SS) equation for two-body system [2]. In the c.m. frame, the SS equation has the form ($\hbar = c = 1$)

$$\left[\sqrt{(-i\vec{\nabla})^2 + m_1^2} + \sqrt{(-i\vec{\nabla})^2 + m_1^2} + V(r)\right] = E\psi(\vec{r}) = 0, \tag{5}$$

where $V(r)$ is the potential (for simplicity we consider separable spherically symmetric potential). It is a problem to find the analytic solution of this equation. The problem originates from two square root operators which cause a serious difficulties. However, it can not be reduced to the second-order differential equation of the Shrödinger type 3.

In this work we study $Q\bar{q}$ mesons and their excitations (resonances) as R2B systems from unified point of view in the framework of the relativistic quantum mechanics (RQM) [23, 24]. Quarkonia as quark-antiquark bound states are simplest among mesons. The quarkonium universal mass formula and "saturating" Regge trajectories were derived in [25] and in [26, 27] applied for gluonia (glueballs). The mass formula was obtained by interpolating between NR heavy $Q\bar{Q}$ quark system and ultra-relativistic limiting case of light $q\bar{q}$ mesons for the Cornell potential [28**?**],

$$V(r) = V_{\rm S}(r) + V_{\rm L}(r) \equiv -\frac{4}{3}\frac{\alpha_{\rm S}}{r} + \sigma r. \tag{6}$$

The short-range Coulomb-type term $V_{\rm S}(r)$, originating from one-gluon exchange, dominates for heavy mesons and the linear one $V_{\rm L}(r)$, which models the string tension, dominates for light mesons. Parameters $\alpha_{\rm S}$ and σ are directly related to basic physical quantities of mesons.

Operators in ordinary quantum mechanics (QM) are Hermitian and the corresponding eigenvalues are real. It is possible to extend the QM Hamiltonian into the complex domain while still retaining the fundamental properties of a quantum theory. One of such approaches is complex quantum mechanics [29]. The complex-scaled method is the extension of theorems and principles proved in QM for Hermitian operators to non-Hermitian operators.

The Cornell potential (6) is a special in hadron physics and results in the complex energy and mass eigenvalues. Separate consideration of two asymptotic components $V_{\rm S}(r)$ and $V_{\rm L}(r)$ of the potential (6) for quarkonia results in the complex-mass expression for resonances, which in the center-of-momentum (c.m.) frame is ($\hbar = c = 1$) [30, 31]:

$$\mathcal{M}_{\rm N}^2 = 4\left[\left(\sqrt{2\sigma\tilde{N}} + \frac{i\tilde{\alpha}m}{N}\right)^2 + \left(m - i\sqrt{2\tilde{\alpha}\sigma}\right)^2\right], \tag{7}$$

where $\tilde{\alpha} = \frac{4}{3}\alpha_{\rm S}$, $\tilde{N} = N+(k+\frac{1}{2})$, $N = k+l+1$, k is radial and l is orbital quantum numbers; it has the form of the squared energy $\mathcal{M}_{\rm N}^2 = 4\left[(\pi_{\rm N})^2 + \mu^2\right]$ of two free relativistic particles with the quarks' complex momenta $\pi_{\rm N}$ and masses μ. This formula allows to calculate in a unified way the centered masses and total widths of heavy and light quarkonia. In our method the energy, momentum and quark masses are *complex*.

A more complicate case are flavored $Q\bar{q}$ mesons. A simplest example of heavy-light two-body system is the hydrogen (H) atom, comprising only a proton and an electron which are stable particles. This simplicity means its properties can be calculated theoretically with impressive accuracy [32]. The spherically symmetric Coulomb potential, with interaction strength parametrized by dimensionless coupling ("fine structure") constant α, is of particular importance in many realms of physics. The H atom can be used as a tool for testing any relativistic two-body theory, because latest measurements for transition frequencies have been determined with a highest precision [33].

III. THE TWO-BODY PROBLEM IN RQM

Standard relativistic approaches for R2B systems run into serious difficulties in solving known relativistic wave equations. The formulation of RQM differs from NR QM by the replacement of invariance under Galilean transformations with invariance under Poincarè transformations. The RQM is also known in the literature as relativistic Hamiltonian dynamics or Poincarè-invariant QM with direct interaction [24]. There are three equivalent forms in the RQM called "instant", "point", and "light-front" forms.

The dynamics of many-particle system in the RQM is specified by expressing ten generators of the Poincarè group, $\hat{M}_{\mu\nu}$ and $\hat{W}_\mu$, in terms of dynamical variables. In the constructing generators for interacting systems it is customary to start with the generators of the corresponding non-interacting system; the interaction is added in the way that is consistent with Poincare algebra. In the relativistic case it is necessary to add an interaction V to more than one generator in order to satisfy the commutation relations of the Poincaré algebra.

The interaction of a relativistic particle with the four-momentum p_μ moving in the external field $A_\mu(x)$ is introduced in QED according to the gauge invariance principle, $p_\mu \to P_\mu = p_\mu - eA_\mu$. The description in the "point" form of RQM implies that the mass operators $\hat{M}^{\mu\nu}$ are the same as for non-interacting particles, i.e., $\hat{M}^{\mu\nu} = M^{\mu\nu}$, and these interaction terms can be presented only in the form of the four-momentum operators $\hat{W}^\mu$ [34].

Consider the R2B problem in classic relativistic theory. Two particles with four-momenta p_1^μ, p_2^μ and the interaction field $W^\mu(q_1,\, q_2)$ together compose a closed conservative system, which can be characterized by the 4-vector $\mathcal{P}^\mu$,

$$\mathcal{P}^\mu = p_1^\mu + p_2^\mu + W^\mu(q_1,\, q_2), \tag{8}$$

where the space-time coordinates q_1^μ, q_2^μ and four-momenta p_1^μ, p_2^μ are conjugate variables, $\mathcal{P}_\mu \mathcal{P}^\mu = \mathsf{M}^2$; here M is the system's invariant mass. Underline, that no external field and each particle of the system can be considered as moving source of the interaction field; the interacting particles and the potential are a unified system. There are the following consequences of (8) and they are key in our approach.

The four-vector (8) describes *free motion* of the bound system and can be presented as,

$$E = \sqrt{\mathbf{p}_1^2 + m_1^2} + \sqrt{\mathbf{p}_2^2 + m_2^2} + W_0(q_1,\, q_2) = \text{const}, \tag{9}$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{W}(q_1,\, q_2) = \text{const}, \tag{10}$$

describing the energy and momentum conservation laws. The energy (9) and total momentum (10) of the system are the constants of motion. By definition, for conservative systems, the integrals (9) and (10) can not depend on time explicitly. This means the interaction $W(q_1,\, q_2)$ should not depend on time, i. e., $W(q_1,\, q_2) => V(\mathbf{r}_1,\, \mathbf{r}_2)$.

It is well known that the potential as a function in 3D-space is defined by the propagator $D(\mathbf{q}^2)$ (Green function) of the virtual particle as a carrier of interaction, where $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2$ is the transfered momentum. In case of the Coulomb potential the propagator is $D(\mathbf{q}^2) = -1/\mathbf{q}^2$; the Fourier transform of $4\pi\alpha D(\mathbf{q}^2)$ gives the Coulomb potential, $V(r) = -\alpha/r$. The relative momentum $\mathbf{q}$ is conjugate to the relative vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, therefore, one can accept that $V(\mathbf{r}_1,\, \mathbf{r}_2) = V(\mathbf{r})$ [35]. If the potential is spherically symmetric, one can write $V(\mathbf{r}) => V(r)$, where $r = |\mathbf{r}|$. Thus, the system's relative time $\tau = t_1 - t_2 = 0$ (instantaneous interaction).

Equations (9) and (10) in the c.m. frame are

$$\mathsf{M} = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + \mathsf{V}(r), \tag{11}$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{W}(\mathbf{r}_1,\, \mathbf{r}_2) = \mathbf{0}, \tag{12}$$

where $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$ that follows from the equality $\mathbf{p}_1 + \mathbf{p}_2 = 0$; this means that $\mathbf{W}(\mathbf{r}_1,\, \mathbf{r}_2) = 0$. The system's mass (11) in the c.m. frame is Lorentz-scalar. In case of free particles ($\mathsf{V} = 0$) the invariant mass $\mathsf{M} = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2}$ can be transformed for $\mathbf{p}^2$ as

$$\mathbf{p}^2 = \frac{1}{4s}(s - m_-^2)(s - m_+^2) \equiv \mathsf{k}^2, \tag{13}$$

which is relativistic invariant, $s = \mathsf{M}^2$ is the Mandelstam's invariant, $m_- = m_1 - m_2$, $m_+ = m_1 + m_2$.

Equation (9) is the zeroth component of the four-vector (8) and the potential W_0 is Lorentz-vector. But, in the c.m. frame the mass (11) is Lorentz-scalar; and what about the potential V? Is it still Lorentz-vector? To show that the potential is Lorentz-scalar, let us reconsider (11) as follows. The relativistic total energy $\epsilon_i(\mathbf{p})$ ($i = 1,\, 2$) of particles in (11) given by $\epsilon_i^2(\mathbf{p}) = \mathbf{p}^2 + m_i^2$ can be represented as sum of the kinetic energy $\tau_i(\mathbf{p})$ and the particle rest mass m_i, i. e., $\epsilon_i(\mathbf{p}) = \tau_i(\mathbf{p}) + m_i$. Then the system's total energy (invariant mass) (11) can be written in the form $\mathsf{M} = \sqrt{\mathbf{p}^2 + \mathsf{m}_1^2(r)} + \sqrt{\mathbf{p}^2 + \mathsf{m}_2^2(r)}$, where

$\mathsf{m}_{1,2}(r) = m_{1,2} + \frac{1}{2}\mathsf{V}(r)$ are the distance-dependent particle masses [36] and (13) with the use of $\mathsf{m}_1(r)$ and $\mathsf{m}_2(r)$ takes the form,

$$\mathbf{p}^2 = K(s)\left[s - (m_+ + \mathsf{V})^2\right] \equiv \mathsf{k}^2 - U(s,\, r), \tag{14}$$

where $K(s) = (s - m_-^2)/4s$, k^2 is squared invariant momentum given by (13) and $U(s,\, r) = K(s)\,[2m_+\mathsf{V} + \mathsf{V}^2]$ is the potential function. The equation (14) is the relativistic analogy of the NR expression $\mathbf{p}^2 = 2\mu[E - V(r)] \equiv \mathsf{k}^2 - U(E, r)$.

The equality (14) with the help of the fundamental correspondence principle gives the two-particle spinless wave equation,

$$\left[\vec{\nabla}^2 + \mathsf{k}^2 - U(s,\, r)\right]\psi(\mathbf{r}) = 0. \tag{15}$$

The equation (15) can not be solved by known methods for the potential (16). Here we use the quasiclassical (QC) method and solve another wave equation [37, 38].

IV. THE INTERACTION POTENTIAL

The NR QM shows very good results in describing bound states; this is partly because the potential is NR concept. In relativistic mechanics one faces with different kind of speculations around the potential, because of absence of a strict definition of the potential in this theory. In NR formulation, the H atom is described by the Schrödinger equation and is usually considered as an electron moving in the external field generated by the proton static electric field given by the Coulomb potential. In relativistic case, the binding energy of an electron in a static Coulomb field (the external electric field of a point nucleus of charge Ze with infinite mass) is determined predominantly by the Dirac eigenvalue [33]. The spectroscopic data are usually analyzed with the use of the Sommerfeld's fine-structure formula [39],

One should note that, in these calculations the S states start to be destroyed above $Z = 137$, and that the P states being destroyed above $Z = 274$. Similar situation we observe from the result of the Klein-Gordon wave equation, which predicts S states being destroyed above $Z = 68$ and P states destroyed above $Z = 82$. Besides, the radial S-wave function $R(r)$ diverges as $r \to 0$. These problems are general for all Lorentz-vector potentials which have been used in these calculations [40, 41]. In general, there are two different relativistic versions: the potential is considered either as the zero component of a four-vector, a Lorentz-scalar or their mixture [42]; its nature is a serious problem of relativistic potential models [43].

This problem is very important in hadron physics where, for the vector-like confining potential, there are no normalizable solutions [43, 44]. There are normalizable solutions for scalar-like potentials, but not for vector-like. This issue was investigated in [25, 40]; it was shown that the effective interaction has to be Lorentz-scalar in order to confine quarks and gluons. The relativistic correction for the case of the Lorentz-vector potential is different from that for the case of the Lorentz-scalar potential [37].

The Cornell potential (6) is fixed by the two free parameters, $\alpha_{\rm S}$ and σ. However, the strong coupling $\alpha_{\rm S}$ in QCD is a function $\alpha_{\rm S}(Q^2)$ of virtuality Q^2 or $\alpha_{\rm S}(r)$ in configuration space. The potential can be modified by introducing the $\alpha_{\rm S}(r)$-dependence, which is unknown. A possible modification of $\alpha_{\rm S}(r)$ was introduced in [26],

$$V_{\rm QCD}(r) = -\frac{4}{3}\frac{\alpha_{\rm S}(r)}{r} + \sigma r, \quad \alpha_{\rm S}(r) = \frac{1}{b_0 \ln[1/(\Lambda r)^2 + (2\mu_g/\Lambda)^2]}, \tag{16}$$

where $b_0 = (33 - 2n_f)/12\pi$, n_f is number of flavors, $\mu_g = \mu(Q^2)$ — gluon mass at $Q^2 = 0$, Λ is the QCD scale parameter. the running coupling $\alpha_S(r)$ in (16) is frozen at $r \to \infty$, $\alpha_\infty = \frac{1}{2}[b_0 \ln(2\mu_g/\Lambda)]^{-1}$, and is in agreement with the asymptotic freedom properties, i. e., $\alpha_S(r \to 0) \to 0$.

V. SOLUTION OF THE QC WAVE EQUATION

Solution of the Shrödinger-type's wave equation (15) can be found by the QC method developed in [38]. In our method one solves the QC wave equation derivation of which is reduced to replacement of the operator $\vec{\nabla}^2$ in (15) by the canonical operator Δ^c without the first derivatives, acting onto the state function $\Psi(\vec{r}) = \sqrt{\det g_{ij}}\psi(\vec{r})$, where g_{ij} is the metric tensor. Thus, instead of (15) one solves the QC equation, for the potential (16),

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2} + \frac{s - m_-^2}{4s}\left[s - \left(m_+ - \frac{4}{3}\frac{\alpha_S(r)}{r} + \sigma r\right)^2\right]\right\}\Psi(\mathbf{r}) = 0. \quad (17)$$

This equation is separated. Solution of the angular equation was obtained in [38] by the QC method in the complex plane, that gives $\mathrm{M}_l = (l + \frac{1}{2})\hbar$, for the angular momentum eigenvalues. These angular eigenmomenta are universal for all spherically symmetric potentials in relativistic and NR cases.

The radial problem has four turning points and cannot be solved by standard methods. We consider the problem separately by the QC method for the short-range Coulomb term (heavy mesons) and the long-range linear term (light mesons). The QC method reproduces the exact energy eigenvalues for all known solvable problems in quantum mechanics [37, 38]. The radial QC wave equation of (17) for the Coulomb term has two turning points and the phase-space integral is found in the complex plane with the use of the residue theory and method of stereographic projection [30, 38] that gives

$$\mathcal{M}_\mathrm{N}^2|_C = \left(\sqrt{\epsilon_\mathrm{N}^2} \pm \sqrt{(\epsilon_\mathrm{N}^2)^*}\right)^2 \equiv \mathrm{Re}\{\epsilon_\mathrm{N}^2\} \pm i\mathrm{Im}\{\epsilon_\mathrm{N}^2\}, \quad (18)$$

where $\epsilon_\mathrm{N}^2 = m_+^2\,(1 - v_\mathrm{N}^2) + 2im_+m_-v_\mathrm{N}$, $v_\mathrm{N} = \frac{2}{3}\alpha_\infty/N$, $N = k + l + 1$.

Large distances in hadron physics are related to the problem of confinement. The radial problem of (17) for the linear term has four turning points, i. e., two cuts between these points. The phase-space integral in this case is found by the same method of stereographic projection as above that results in the cubic equation [36]: $s^3 + a_1 s^2 + a_2 s + a_3 = 0$, where $a_1 = 16\tilde{\alpha}_\infty\sigma - m_-^2$, $a_2 = 64\sigma^2\left(\tilde{\alpha}_\infty^2 - \tilde{N}^2 - \tilde{\alpha}_\infty m_-^2/4\sigma\right)$, $a_3 = -(8\tilde{\alpha}_\infty\sigma m_-)^2$, $\tilde{N} = N + k + \frac{1}{2}$, $\tilde{\alpha}_\infty = \frac{4}{3}\alpha_\infty$, $\alpha_\infty = \alpha_S(r \to \infty)$. The first root $s_1(N)$ of this equation gives the physical solution (complex eigenmasses), $\mathsf{M}_1^2|_L = s_1(N)$, for the squared invariant mass.

Two exact asymptotic solutions obtained such a way are used to derive the interpolating mass formula. The interpolation procedure for these two solutions [25] is used to derive the meson's complex-mass formula,

$$\mathcal{M}_\mathrm{N}^2 = (m_1 + m_2)^2\left(1 - v_\mathrm{N}^2\right) \pm 2im_+m_-v_\mathrm{N} + \mathsf{M}_1^2|_L. \quad (19)$$

The real part of the square root of (19) defines the centered masses and its imaginary part defines the total widths, $\Gamma_\mathrm{N}^\mathrm{TOT} = -2\,\mathrm{Im}\{\mathsf{M}_\mathrm{N}\}$, of mesons and resonances [30, 31].

In the QC method not only the total energy, but also momentum of a particle-wave in bound state is the *constant of motion.* Solution of the QC wave equation in the whole region is written in elementary functions as [38]

$$\mathsf{R}(r) = C_n \begin{cases} \frac{1}{\sqrt{2}} e^{|\mathsf{k}_n| r - \phi_1}, & r < r_1, \\ \cos(|\mathsf{k}_n| r - \phi_1 - \frac{\pi}{4}), & r_1 \le r \le r_2, \\ \frac{(-1)^n}{\sqrt{2}} e^{-|\mathsf{k}_n| r + \phi_2}, & r > r_2, \end{cases} \tag{20}$$

where $C_n = \sqrt{2|p_n|/[\pi(n+\frac{1}{2})+1]}$ is the normalization coefficient, k_n is the corresponding eigenmomentum found from solution of (15), $\phi_1 = -\pi(n+\frac{1}{2})/2$ and $\phi_2 = \pi(n+\frac{1}{2})/2$ are the values of the phase-space integral at the turning points r_1 and r_2, respectively.

The free fit to the data [45] shows a good agreement for the light and heavy $Q\bar{q}$ meson and their resonances. To demonstrate efficiency of the model we calculate the leading-state masses of the ρ and B^* meson resonances (see tables, where masses are in MeV). Note, that the gluon mass, $m_g = 416\,\mathrm{MeV}$, and the string tension $\sigma = 140\,\mathrm{MeV}^2$ in the independent fitting are the same. Note, that gluon mass μ_g is the same for glueballs [26].

TABLE I. The masses of the $\rho^{\pm}$-mesons and resonances

Meson	J^{PC}	E_n^{ex}	E_n^{th}	Parameters in (19)
$\rho\ (1S)$	1^{--}	776	776	$\Lambda = 488$ MeV
$a_2(1P)$	2^{++}	1318	1315	$\mu_g = 416\,\mathrm{MeV}$
$\rho_3(1D)$	3^{--}	1689	1689	$\sigma = 140\,\mathrm{MeV}^2$
$a_4(1F)$	4^{++}	1996	1993	$m_d = 119\,\mathrm{MeV}$
$\rho\ (1G)$	5^{--}		2257	$m_u = 69\,\mathrm{MeV}$
$\rho\ (2S)$	1^{--}	1720	1688	
$\rho\ (2P)$	2^{++}		1993	
$\rho\ (2D)$	3^{--}		2257	

TABLE II. The masses of the B^{*0}-mesons and resonances

Meson	J^{PC}	E_n^{ex}	E_n^{th}	Parameters in (19)
$B^*(1S)$	1^{--}	5325	5325	$\Lambda = 75\,\mathrm{MeV}$
$B_2^*(1P)$	2^{++}	5743	5743	$m_g = 416\,\mathrm{MeV}$
$B_3^*(1D)$	3^{--}		5946	$\sigma = 140\,\mathrm{MeV}^2$
$B_4^*(1F)$	4^{++}		6088	$m_b = 2856\,\mathrm{MeV}$
$B_5^*(1G)$	5^{--}		6205	$m_d = 25\,\mathrm{MeV}$
$B^*(2S)$	1^{--}		5834	
$B^*(2P)$	2^{++}		6034	
$B^*(2D)$	3^{--}		6175	

The d quark effective mass decreases if heavy quark mass increases.

CONCLUSION

We have considered bound states as relativistic bound systems in the potential approach without using the QCD BS equation or its reductions. We have modeled mesons containing light and heavy quarks and their resonances in the framework of RQM. We have began our investigation within relativistic classical mechanics using the basic principles of symmetries, i.e., the energy and momentum conservations' laws in Minkowskiy space. The potential of interaction was selected to be the Lorentz-scalar function of the spatial variable r. The concept of position dependent particle mass was used. Using the correspondence principle, we have deduced, from the R2B classic equation, the two-particle wave equation.

We have calculated masses of light-heavy $S = 1$ mesons containing d quark and their resonances, i. e., $\rho^{\pm}$ and B^{*0} states for universal string tension $\sigma = 140\,\mathrm{MeV}^2$, which can be considered as a fundamental constant in hadron physics. We have shown that quarks inside the system move as free particles. Using the complex-mass analysis, we have derived the meson interpolating masses formula (19), in which the real and imaginary parts are exact expressions. This approach allows to simultaneously describe in the unified way the centered masses and total widths of resonances. We have shown here the calculation results for unflavored light and beauty mesons and resonances, however, other mesons, containing strange and charm quarks can be described also well.

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