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Massive and Massless Fields with Spin 3/2, Solutions of the Wave Equation and Helicity Operator

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In the paper, solutions in the form of plane waves for a massive spin 3/2 particle are examined. The wave equation gives 4 algebraic equations for 8 unknown variables, which assumes existence of 4 independent solutions. In order to relate the choice of independent solutions to quantum number of a physical operator, we study the problem of eigenvectors for relevant helicity operator. As expected, we get 4 eigenvalues, $\sigma = \pm 1/2, \pm 3/2$. The values $\sigma = \pm 1/2$ turn out to have double multiplicity, this leads to existence of two different eigenstates both for $\sigma = -1/2$ and $\sigma = +1/2$. It is shown that the states with the values represent exact solutions of the wave equation. However, the double degenerate states separately do not. It is shown that exact solutions of the wave equation can be constructed in the form of special linear combinations of those. Thus, there constructed a complete system of exact solution for a massive spin 3/2 particle in momentum-helicity basis.

Initial wave equation for vector bispinor $\Psi_a(x)$, describing a massless spin 3/2 particle in Rarita-Sscwinger form, is transformed to a new basis $\tilde{\Psi}_a(x)$, in which the gauge symmetry in the theory becomes evident: there exist solutions in the form of 4-gradient of an arbitrary bispinor $\tilde{\Psi}^0_a(x) = \partial_a \Psi(x)$. For 16-component equation in this new basis, two independent solution are constructed explicitly, which do not contain gauge constituents. Previously, in the basis $\Psi_a(x)$, the eigenvalue problem for helicity operator of the spin 3/2 particle was solved, and six types of eigenstates were found; there are possible eigenvalues $\sigma = \pm 1/2, \pm 3/2$; the states with $\sigma = \pm 1/2$ are doubly degenerate. Massless solutions are transformed to initial Rarita-Schwinger basis, after that they are decomposed into linear combinations of helicity states, the relevant formulas contain terms related to helicities $\sigma = \pm 1/2$ and $\sigma = \pm 3/2$.

Key words: zero mass particle, spin 3/2, Rarita-Schwinger basis, gauge symmetry, plane wave solutions, massless solutions and helicity

I. INTRODUCTION

The theory of spin 3/2 particle is attracted steady interest after the seminal investigation by Pauli and Fierz - see [1-15]. Let us recall the most significant aspect of spin 3/2 particle theory. First of all, it is the problem of choosing an initial system of equations. The most consistent is an approach based on Lagrangian formalism and a correct first order equation for multi-component wave function which are based on the general theory of 1-st order relativistic wave equations. However investigations are based on the use of 2-nd order equations. Such an choice is of prime importance when we take into account the present of external electromagnetic (or gravitational) fields. Applying the first order approach ensures correct solving the problem of independent degreee of freedom in presence of external fields; for instance see in [16]). The great attention was given to existence in this theory solutions which correspond of a particle moving with velocity greater than the light velocity. Finally a separate interest has a massless case for spin 3/2 field, when – as shown by Pauli and Fierz – there exist specific gauge symmetry: the 4-gradient of arbitrary bispinor function $\Phi(x)$ provides us with solution for the massless field equation for instance, see in [16]).

The main goal of the present paper is to follow the problem of degrees of freedom for massive and massless 3/2 particle and the role of helicity operator, when constructing plane wave solutions. As basic we use Rarita – Schwinger formalism.

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II. ON DESCRIBING THE MASSLESS SPIN 3/2 FIELD

In Rarita – Schwinger approach, the wave equation for massive spin 3/2 particle take the form

$$(\gamma^a \partial_a + iM)\Psi_c - \frac{1}{3}(\gamma^b \partial_c + \gamma_c \partial^b)\Psi_b + \frac{1}{3}\gamma_c(\gamma^a \partial_a - iM)\gamma^b \Psi_b = 0 \,. \tag{1}$$

the wave function is vector-bispinor under the Lorentz group. Below we need several formulas for Dirac matrices:

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2g^{ab}, \quad \gamma^a \gamma_a = 4 \,, \quad \gamma^a \gamma^b \gamma^d = \gamma^a g^{bd} - \gamma^b g^{ad} + \gamma^d g^{ab} + i\gamma^5 \epsilon^{abdc} \gamma_c \,, \tag{2}$$

Levi-Civira tensor is specified as ϵ^{cabd}, $\epsilon^{0123} = +1$. Starting from eq. (1) one can derive some additional constraints on $\Psi_a(x)$. So, multiplying eq. (1) by the matrix γ^c, we get

$$\partial_b \, \Psi^b = \frac{iM}{2} \gamma_b \Psi^b, \tag{3}$$

it is the first constraint. Now, let us act on eq. (1) ny operator ∂^c, further with the use of the above formulas (2) we obtain

$$iM \, \partial^c \Psi_c + \gamma^a \partial_a (\frac{2}{3} \partial^c \Psi_c - \frac{iM}{3} \, \gamma^c \Psi_c) = 0 \,, \tag{4}$$

it is the second constraint. Whence with the the first condition (3) in mind, it follows

$$\partial^c \Psi_c = 0 \quad (M \neq 0) \,. \tag{5}$$

therefore, the first constraint gives $\gamma_b \Psi^b = 0$, Taking into account two last restrictions, instead of initial eq. (1) we arrive at an equivalent system

$$(i\gamma^a \partial_a - M)\Psi_c = 0 \,, \quad \partial^c \Psi_c = 0, \quad \gamma^c \Psi_c = 0 \,. \tag{6}$$

The case of massless particle is substantially different. From the very beginning, in eq. (1) we set $M = 0$:

$$\gamma^a \partial_a \Psi_c - \frac{1}{3}(\gamma^b \partial_c + \gamma_c \partial^b)\Psi_b + \frac{1}{3}\gamma_c \gamma^a \partial_a \gamma^b \Psi_b = 0 \,. \tag{7}$$

Now, the first constraint reads as

$$\partial_b \, \Psi^b = 0 \,. \tag{8}$$

The second condition (3) takes the form $\gamma^a \partial_a \partial^c \Psi_c = 0$, evidently it does not add restrictions to the first constraint (8). Therefore, for massless case eq. (7) with (8) in mind may be presented as follows:

$$\gamma^a \partial_a \Psi_c - \frac{1}{3}\gamma^b \partial_c \Psi_b + \frac{1}{3}\gamma_c \gamma^a \gamma^b \partial_a \Psi_b = 0 \,, \quad \partial_b \Psi^b = 0 \,. \tag{9}$$

Further we use the wave equation (7). It can be transformed to the form, when existence of gauge solutions of gradient type becomes evident. To this end, we re-write eq. (7) in matrix form

$$\Gamma^a \partial_a \Psi = 0 \,, \quad \Psi = (\Psi_l) \,, \tag{10}$$

where 16×16 matrices Γ^a are given by the formula

$$(\Gamma^a)_k{}^l = \gamma^a \, \delta^l_k - \frac{1}{3}\gamma^l \delta^a_k - \frac{1}{3}\gamma_k g^{al} + \frac{1}{3}\gamma_k \gamma^a \gamma^l . \tag{11}$$

Let us make two transformations on eq. (10): first multiply it by a matrix C with nonvanishing determinant, and then introduce new wave function by a linear transformation S:

$$\Gamma^a \implies \Gamma^{'a} = C \, \Gamma^a \quad \implies \quad \tilde{\Gamma}^a = \, S \, \Gamma^{'a} \, S^{-1} \,, \ \tilde{\Psi} = S \, \Psi \,. \tag{12}$$

these two matrices will be specified below. Let the matrices C S have the structure:

$$\begin{aligned} C_a{}^b &= \delta^b_a + c\gamma_a \gamma^b \,, \quad S_a{}^b = \delta^b_a + a\gamma_a \gamma^b \,, \\ (S^{-1})_a{}^b &= \delta^b_a + b\gamma_a \gamma^b \,, \quad a + b + 4ab = 0 \,. \end{aligned} \tag{13}$$

where $a,\ b,\ c$ are some numerical parameters; constraint on a and b follows from condition $S\,S^{-1}=I$. In accordance with (12) and (13), we first find Γ'^{a}:

$$(\Gamma'^{a})_{k}{}^{l} = [\ \gamma^{a}\ \delta^{l}_{k}\ -\ \frac{1}{3}\ \gamma^{l}\ \delta^{a}_{k}\ + (2c\ -\frac{1}{3})\ \gamma_{k}\ g^{al}\ +\frac{1}{3}\ \gamma_{k}\ \gamma^{a}\ \gamma^{l}]\ ,$$

a then – matrices $\tilde{\Gamma}^{a}$:

$$(\tilde{\Gamma}^{a})_{k}{}^{l} = \gamma^{a}\delta^{l}_{\rho}\left\{1-\left[\frac{b+1}{3}+b(\frac{2c-1}{3}(1+4a)+2a)\right]\right\}+$$
$$+\gamma^{l}\delta^{a}_{k}\left\{\frac{2b-1}{3}+\left[\frac{b+1}{3}+b(\frac{2c-1}{3}(1+4a)+2a)\right]\right\}+$$
$$+\gamma_{k}g^{al}\left\{\left[(2c-1)\frac{1+4a}{3}+2a\right]+\left[\frac{b+1}{3}+b((2c-1)\frac{1+4a}{3}+2a)\right]\right\}+$$
$$+i\gamma^{5}\epsilon^{als}_{k}\gamma_{s}\left[\frac{b+1}{3}+b((2c-1)\frac{1+4a}{3}+2a)\right]\ . \qquad (14)$$

let us require that in $\tilde{\Gamma}^{a}$ all term except the one containing the Levy-Civita tensor be vanish, this results in the following equations:

$$a+b+4ab=0\,,\ \ 1-\left[\frac{b+1}{3}+b((2c-1)\frac{1+4a}{3}+2a)\right]=0\,,$$
$$\frac{2b+1}{3}+\left[\frac{b+1}{3}+b((2c-1)\frac{1+4a}{3}+2a)\right]=0\,,$$
$$(1+4a)\frac{2c-1}{3}+2a+\left[\frac{b+1}{3}+b((2c-1)\frac{1+4a}{3}+2a)\right]=0\,.$$

It is readily verified that its solution is given as

$$a=-\frac{1}{3}\,,\quad b=-1\,,\quad c=+2\,. \qquad (15)$$

Thus, we find the needed linear transformation

$$\tilde{\Psi}_{k}=S_{k}{}^{l}\Psi_{l},\quad S_{k}{}^{l}=\delta^{l}_{k}-\frac{1}{3}\ \gamma_{k}\gamma^{l}\,, \qquad (16)$$

and new matrices $\tilde{\Gamma}^{a}$ have the structure $(\tilde{\Gamma}^{a})_{k}{}^{l}\ =+i\ \gamma^{5}\epsilon_{k}{}^{als}\gamma_{s}$. Therefore, we arrive at the equivalent form of the wave equation for massless spin 3/2 particle:

$$(\tilde{\Gamma}^{a})_{l}{}^{k}\tilde{\Psi}_{l}(x)=0\quad\Longrightarrow\quad i\ \gamma^{5}\epsilon_{k}{}^{nal}\gamma_{n}\partial_{a}\tilde{\Psi}_{l}(x)=0\ ; \qquad (17)$$

in eq. (17) we may omit the multiplier $i\gamma^{5}$. From eq. (17) follows that the vector-bispinor $\tilde{\Psi}^{(0)}_{l}$ being the gradient of arbitrary bispinor $\varphi(x)$:

$$\tilde{\Psi}^{(0)}_{l}(x)=\partial_{l}\varphi(x) \qquad (18)$$

provides us with solution of eq. (17). This property is called as gauge symmetry in massless spin 3/2 field theory. In initial basis, such gradien=-type solution are presented by the formula

$$\Psi^{(0)}_{l}(x)=(\delta_{l}{}^{k}-\gamma_{l}\gamma^{k})\partial_{k}\varphi(x)\,. \qquad (19)$$

Note that explicit form of eq. (18) is rather cumbersome:

$$\begin{aligned}
&\gamma_{1}(\partial_{2}\tilde{\Psi}_{3}-\partial_{3}\tilde{\Psi}_{2})+\gamma_{2}(\partial_{3}\tilde{\Psi}_{1}-\partial_{1}\tilde{\Psi}_{3})+\gamma_{3}(\partial_{1}\tilde{\Psi}_{2}-\partial_{2}\tilde{\Psi}_{1})=0\,,\\
&-\gamma_{0}(\partial_{2}\tilde{\Psi}_{3}-\partial_{3}\tilde{\Psi}_{2})+\partial_{0}(\gamma_{2}\tilde{\Psi}_{3}-\gamma_{3}\tilde{\Psi}_{2})-(\gamma_{2}\partial_{3}-\gamma_{3}\partial_{2})\tilde{\Psi}_{0}=0\,,\\
&-\gamma_{0}(\partial_{3}\tilde{\Psi}_{1}-\partial_{1}\tilde{\Psi}_{3})+\partial_{0}(\gamma_{3}\tilde{\Psi}_{1}-\gamma_{1}\tilde{\Psi}_{3})-(\gamma_{3}\partial_{1}-\gamma_{1}\partial_{3})\tilde{\Psi}_{0}=0\,,\\
&-\gamma_{0}(\partial_{1}\tilde{\Psi}_{2}-\partial_{2}\tilde{\Psi}_{1})+\partial_{0}(\gamma_{1}\tilde{\Psi}_{2}-\gamma_{2}\tilde{\Psi}_{1})-(\gamma_{1}\partial_{2}-\gamma_{2}\partial_{1})\tilde{\Psi}_{0}=0\,.
\end{aligned} \qquad (20)$$

III. SEPARATING THE VARIABLES

In Rarita – Schwinger approach, the free spin 3/2 particle (in massive case) may be described by the set of three equations

$$(i\gamma^a \partial_a - \frac{mc}{\hbar}) \, \Psi_l(x) = 0 \, , \quad \gamma^l \Psi_l(x) = 0 \, , \;\; \partial_l \Psi^l(x) = 0 \, , \tag{21}$$

where 16-component wave function $\Psi_l(x)$ is a vector-bispinor with respect to Lorentz group:

$$\Psi_{A(l)}(x) = \left| \begin{array}{cccc} \Psi_{1(0)}(x) & \Psi_{1(1)}(x) & \Psi_{1(2)}(x) & \Psi_{1(3)}(x) \\ \Psi_{2(0)}(x) & \Psi_{2(1)}(x) & \Psi_{2(2)}(x) & \Psi_{2(3)}(x) \\ \Psi_{3(0)}(x) & \Psi_{3(1)}(x) & \Psi_{3(2)}(x) & \Psi_{3(3)}(x) \\ \Psi_{4(0)}(x) & \Psi_{4(1)}(x) & \Psi_{4(2)}(x) & \Psi_{4(3)}(x) \end{array} \right| , \tag{22}$$

A and (l) designate bispinor and vector indices respectively Wee use the Dirac matrices in spinor basis:

$$\gamma^0 = \left| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right| , \gamma^1 = \left| \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right| , \gamma^2 = \left| \begin{array}{cccc} 0 & 0 & 0 & +i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ +i & 0 & 0 & 0 \end{array} \right| , \gamma^3 = \left| \begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right|$$

Solutions od eqs. (21) are searched in form of planes waves:

$$\Psi_l(x) = e^{-i\epsilon t/\hbar} e^{+ip_k x^k} A_l, \qquad A_l = \left| \begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{array} \right| . \tag{23}$$

We use the quantities$mc/\hbar = M, \epsilon/\hbar c = E, p_i/\hbar = k_j$. From eqs. (21) follow three algebraic equations (remembering the rules, $A^0 = A_0, A^j = -A_j$)

$$(\gamma^0 E - \gamma^j k_j - M) \, A_l = 0 \, , \quad \gamma^0 A_0 + \gamma^j A_j = 0 \, , \quad E \, A_0 + k_j \, A_j = 0 \, . \tag{24}$$

The first (main) equation in (24) takes the form

$$\left| \begin{array}{cccc} -M & 0 & (E+k_3) & (k_1 - ik_2) \\ 0 & -M & (k_1 + ik_2) & (E - k_3) \\ (E-k_3) & -(k_1 - ik_2) & -M & 0 \\ -(k_1 + ik_2) & (E+k_3) & 0 & -M \end{array} \right| \left| \begin{array}{c} a_l \\ b_l \\ c_l \\ d_l \end{array} \right| = 0, \quad l = 0,1,2,3 \, ; \tag{25}$$

The determinant of the system must be equal to zero:

$$(-M^2 + E^2 - k_3^2 - k_2^2 - k_1^2)^2 = 0 \implies E^2 = M^2 + \mathbf{k}^2. \tag{26}$$

It is readily proved that determinant of 3×3-matrix vanishes too:

$$\det \left| \begin{array}{ccc} -M & 0 & (E+k_3) \\ 0 & -M & (k_1 + ik_2) \\ (E-k_3) & -(k_1 - ik_2) & -M \end{array} \right| = -M(M^2 - E^2 + \mathbf{k}^2) = 0 \, ,$$

This mens that the rank of the system (25) equals to 2. Therefore, in system (25) only two equations are independent, let these be the first two

$$(E + k_3) \, c_l + (k_1 - ik_2) \, d_l = M \, a_l \, , \;\; (k_1 + ik_2) \, c_l + (E - k_3) \, d_0 = M \, b_l \, . \tag{27}$$

Its solution is (recall that $l = 0,1,2,3$)

$$c_l = \frac{E - k_3}{M} \, a_l - \frac{k_1 - ik_2}{M} \, b_l \, , \quad d_l = -\frac{k_1 + ik_2}{M} \, a_l + \frac{E + k_3}{M} \, b_l \, , \tag{28}$$

or shortly

$$c_l = \alpha a_l + \beta b_l, \; d_l = \rho a_l + \gamma b_l \, . \tag{29}$$

The second equation in (24) leads to

$$a_0 + a_3 = -(b_1 - ib_2)\,, \quad b_0 - b_3 = -(a_1 + ia_2)\,,$$
$$c_0 - c_3 = d_1 - id_2\,, \quad d_0 + d_3 = c_1 + ic_2\,. \tag{30}$$

Taking in mind (29), one excludes the variables c_l, d_l in (30):

$$a_0 + a_3 = -(b_1 - ib_2)\,, \qquad b_0 - b_3 = -(a_1 + ia_2)\,,$$
$$\alpha(a_0 - a_3) + \beta(b_0 - b_3) = \rho(a_1 - ia_2) + \gamma(b_1 - ib_2)\,,$$
$$\rho(a_0 + a_3) + \gamma(b_0 + b_3) = \alpha(a_1 + ia_2) + \beta(b_1 + ib_2)\,. \tag{31}$$

In (31) we have 4 linear constraints on 8 variables, a_l, b_l.
Finally, the third equation in (24) yields

$$Ea_0 + k_1a_1 + k_2a_2 + k_3a_3 = 0\,, \quad Eb_0 + k_1b_1 + k_2b_2 + k_3b_3 = 0\,,$$
$$Ec_0 + k_1c_1 + k_2c_2 + k_3c_3 = 0\,, \quad Ed_0 + k_1d_1 + k_2d_2 + k_3d_3 = 0\,. \tag{32}$$

We immediately note that two last equations in (32), if one takes into account (28), become consequences of two first equations in (32). : Therefore, in the system (32) we may preserve only two independent relations

$$Ea_0 + k_1a_1 + k_2a_2 + k_3a_3 = 0\,, \quad Eb_0 + k_1b_1 + k_2b_2 + k_3b_3 = 0\,. \tag{33}$$

Let us collect equations (31) and (33) together:

$$a_0 + a_3 = -(b_1 - ib_2)\,, \qquad b_0 - b_3 = -(a_1 + ia_2)\,,$$
$$\alpha(a_0 - a_3) + \beta(b_0 - b_3) = \rho(a_1 - ia_2) + \gamma(b_1 - ib_2)\,,$$
$$\rho(a_0 + a_3) + \gamma(b_0 + b_3) = \alpha(a_1 + ia_2) + \beta(b_1 + ib_2)\,,$$
$$Ea_0 + k_1a_1 + k_2a_2 + k_3a_3 = 0\,,\ Eb_0 + k_1b_1 + k_2b_2 + k_3b_3 = 0\,. \tag{34}$$

It is readily checked that from two first equations in (34) follows two last. Indeed, first by exclusion method we derive equations containing respectively the variables a_l and b_l (remembering explicit form of coefficients $\alpha, \beta, \rho, \sigma$):

$$(E - k_3)(a_0 - a_3) + (k_1 + ik_2)(a_1 - ia_2) = -(k_1 - ik_2)(a_1 + ia_2) - (E + k_3)(a_0 + a_3)\,,$$
$$(k_1 + ik_2)(b_1 - ib_2) + (E - k_3)(b_0 - b_3) = -(E + k_3)(b_0 + b_3) - (k_1 - ik_2)(b_1 + ib_2)\,, \tag{35}$$

they give

$$Ea_0 + k_1a_1 + k_2a_2 + k_3a_3 = 0\,, \quad Eb_0 + k_1b_1 + k_2b_2 + k_3b_3 = 0\,. \tag{36}$$

Therefore, in system (34) we may ignore two last equations. Turning to (**??**), with the help of two first equations

$$a_0 = -a_3 - (b_1 - ib_2)\,, \qquad b_0 = +b_3 - (a_1 + ia_2) \tag{37}$$

we may exclude the variables a_0, b_0, this yields

$$k_1a_1 + k_2a_2 + k_3a_3 = +Ea_3 + E(b_1 - ib_2)\,, \quad k_1b_1 + k_2b_2 + k_3b_3 = -Eb_3 + E(a_1 + ia_2)\,. \tag{38}$$

In (38) we have 2 equations for 6 variables, so there should exist 4 independent solutions.
It is readily proved that taking in mind (37), from (38) follow eqs. (36). This means that eqs. (38) are equivalent to following 4 equations

$$Ea_0 + k_1a_1 + k_2a_2 + k_3a_3 = 0\,, \quad Eb_0 + k_1b_1 + k_2b_2 + k_3b_3 = 0\,.$$
$$a_0 + a_3 = -(b_1 - ib_2)\,, \quad b_0 - b_3 = -(a_1 + ia_2)\,. \tag{39}$$

In order to relate 4 independent solutions of system (39) to quantum number of some physical operators, in the next section we will study the problem of eigenstates of the helicity operator for of vector-bispinor plane waves.

IV. HELICITY OPERATOR

On vector-bispinor plane waves let us diagonalize helicity operator Σ:

$$\Sigma = -i\,(S_1\partial_1 + S_2\partial_2 + S_3\partial_3)\,, \qquad \Sigma\,\Psi(x) = \sigma\,\Psi(x)\,. \tag{40}$$

Taking in mind substitution in the plane wave form we find the followin representation for helicity operator

$$\Sigma = i\,(k_1\sigma^{23} + k_2\sigma^{31} + k_3\sigma^{12}) \otimes I + I \otimes i\,(k_1 j^{23} + k_2 j^{31} + k_3 j^{12})\,. \tag{41}$$

We need expressions for the matrices:

$$\sigma^{23} = \frac{1}{2}\gamma^2\gamma^3 = -\frac{i}{2}\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \sigma^{31} = \frac{1}{2}\gamma^3\gamma^1 = -\frac{i}{2}\begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{vmatrix}, \sigma^{12} = \frac{1}{2}\gamma^1\gamma^2 = -\frac{i}{2}\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix},$$

$$j^{23} = \delta^2_k g^{3l} - \delta^3_k g^{2l} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, j^{31} = \delta^3_k g^{1l} - \delta^1_k g^{3l} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, j^{12} = \delta^1_k g^{2l} - \delta^2_k g^{1l} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

With (**??**)–(**??**) in mind, for operator Σ we find

$$\Sigma = \frac{1}{2}\begin{vmatrix} k_3 & k_1 - ik_2 & 0 & 0 \\ k_1 + ik_2 & -k_3 & 0 & 0 \\ 0 & 0 & k_3 & k_1 - ik_2 \\ 0 & 0 & k_1 + ik_2 & -k_3 \end{vmatrix} \otimes I + I \otimes i \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -k_3 & k_2 \\ 0 & k_3 & 0 & -k_1 \\ 0 & -k_2 & k_1 & 0 \end{vmatrix}. \tag{42}$$

To clarify the action of Σ on the wave function, it suffice to recall the action og generators on vector-bispinor:

$$[\delta_{AB} + \delta\omega_{12}\; i(\sigma^{12})_{AB}]\;[\delta_{ln} + \delta\omega_{12}\; i(j^{12})_{ln}]\Psi_{Bn} = \Psi_{Al} + \delta\omega_{12}\;\{\; i(\sigma^{12})_{AB}\Psi_{Bl} + i(j^{12})_{ln}\Psi_{An}(\tilde{j^{12}})_{nl}\;\}\;;$$

where the symbol $\tilde{S}$ over S designates a transposed matrix.

Taking in mind identity

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{vmatrix}\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & +k_3 & -k_2 \\ 0 & -k_3 & 0 & +k_1 \\ 0 & +k_2 & -k_1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & (-k_3a_2 + k_2a_3) & (k_3a_1 - k_1a_3) & (-k_2a_1 + k_1a_2) \\ 0 & (-k_3b_2 + k_2b_3) & (k_3b_1 - k_1b_3) & (-k_2b_1 + k_1b_2) \\ 0 & (-k_3c_2 + k_2c_3) & (k_3c_1 - k_1c_3) & (-k_2c_1 + k_1c_2) \\ 0 & (-k_3d_2 + k_2d_3) & (k_3d_1 - k_1d_3) & (-k_2d_1 + k_1d_2) \end{vmatrix}$$

we reduce eigenvalue equation $\Sigma\,\Psi = \sigma\,\Psi$ to the form

$$\frac{1}{2}\begin{vmatrix} k_3 & (k_1 - ik_2) & 0 & 0 \\ (k_1 + ik_2) & -k_3 & 0 & 0 \\ 0 & 0 & k_3 & (k_1 - ik_2) \\ 0 & 0 & (k_1 + ik_2) & -k_3 \end{vmatrix}\begin{vmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{vmatrix} = \sigma \begin{vmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{vmatrix},$$

$$\frac{1}{2}\begin{vmatrix} k_3 & (k_1 - ik_2) & 0 & 0 \\ (k_1 + ik_2) & -k_3 & 0 & 0 \\ 0 & 0 & k_3 & (k_1 - ik_2) \\ 0 & 0 & (k_1 + ik_2) & -k_3 \end{vmatrix}\begin{vmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{vmatrix} + i\begin{vmatrix} (k_2a_3 - k_3a_2) \\ (k_2b_3 - k_3b_2) \\ (k_2c_3 - k_3c_2) \\ (k_2d_3 - k_3d_2) \end{vmatrix} = \sigma \begin{vmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{vmatrix},$$

$$\frac{1}{2}\begin{vmatrix} k_3 & (k_1 - ik_2) & 0 & 0 \\ (k_1 + ik_2) & -k_3 & 0 & 0 \\ 0 & 0 & k_3 & (k_1 - ik_2) \\ 0 & 0 & (k_1 + ik_2) & -k_3 \end{vmatrix}\begin{vmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{vmatrix} + i\begin{vmatrix} (k_3a_1 - k_1a_3) \\ (k_3b_1 - k_1b_3) \\ (k_3c_1 - k_1c_3) \\ (k_3d_1 - k_1d_3) \end{vmatrix} = \sigma \begin{vmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{vmatrix},$$

$$\frac{1}{2}\begin{vmatrix} k_3 & (k_1 - ik_2) & 0 & 0 \\ (k_1 + ik_2) & -k_3 & 0 & 0 \\ 0 & 0 & k_3 & (k_1 - ik_2) \\ 0 & 0 & (k_1 + ik_2) & -k_3 \end{vmatrix}\begin{vmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{vmatrix} + i\begin{vmatrix} (k_1a_2 - k_2a_1) \\ (k_1b_2 - k_2b_1) \\ (k_1c_2 - k_2c_1) \\ (k_1d_2 - k_2d_1) \end{vmatrix} = \sigma \begin{vmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{vmatrix}.$$

Whence follow the 16 linear equations:

$$(k_3 - 2\sigma)\ a_0 + (k_1 - ik_2)\ b_0 = 0\,,$$
$$(k_1 + ik_2)\ a_0 - (k_3 + 2\sigma)\ b_0 = 0\,;$$

$$(k_3 - 2\sigma)\ c_0 + (k_1 - ik_2)\ d_0 = 0\,,$$
$$(k_1 + ik_2)\ c_0 - (k_3 + 2\sigma)\ d_0 = 0\,,$$

$$(k_3 - 2\sigma)\ a_1 + (k_1 - ik_2)\ b_1 + 2ik_2\ a_3 - 2ik_3\ a_2 = 0\,;$$
$$(k_1 + ik_2)\ a_1 - (k_3 + 2\sigma)\ b_1 + 2ik_2\ b_3 - 2ik_3\ b_2 = 0\,,$$

$$(k_3 - 2\sigma)\ c_1 + (k_1 - ik_2)\ d_1 + 2ik_2\ c_3 - 2ik_3\ c_2 = 0\,,$$
$$(k_1 + ik_2)\ c_1 - (k_3 + 2\sigma)\ d_1 + 2ik_2\ d_3 - 2ik_3\ d_2 = 0\,;$$

$$(k_3 - 2\sigma)\ a_2 + (k_1 - ik_2)\ b_2 + 2ik_3\ a_1 - 2ik_1\ a_3 = 0\,,$$
$$(k_1 + ik_2)\ a_2 - (k_3 + 2\sigma)\ b_2 + 2ik_3\ b_1 - 2ik_1\ b_3 = 0\,,$$

$$(k_3 - 2\sigma)\ c_2 + (k_1 - ik_2)\ d_2 + 2ik_3\ c_1 - 2ik_1\ c_3 = 0\,,$$
$$(k_1 + ik_2)\ c_2 - (k_3 + 2\sigma)\ d_2 + 2ik_3\ d_1 - 2ik_1\ d_3 = 0\,;$$

$$(k_3 - 2\sigma)\ a_3 + (k_1 - ik_2)\ b_3 + 2ik_1\ a_2 - 2ik_2\ a_1 = 0\,,$$
$$(k_1 + ik_2)\ a_3 - (k_3 + 2\sigma)\ b_3 + 2ik_1 b_2 - 2ik_2 b_1 = 0\,,$$

$$(k_3 - 2\sigma)\ c_3 + (k_1 - ik_2)\ d_3 + 2ik_1\ c_2 - 2ik_2\ c_1 = 0\,,$$
$$(k_1 + ik_2)\ c_3 - (k_3 + 2\sigma)\ d_3 + 2ik_1\ d_2 - 2ik_2\ d_1 = 0\,.$$

In the whole system we can see 4 unlinked subsystems, $16 = 2 + 2 + 6 + 6$.

The first two subsystems

$$\begin{array}{c} (k_3 - 2\sigma)\ a_0 + (k_1 - ik_2)\ b_0 = 0\,, \quad (k_1 + ik_2)\ a_0 - (k_3 + 2\sigma)\ b_0 = 0\ ; \\ (k_3 - 2\sigma)\ c_0 + (k_1 - ik_2)\ d_0 = 0\,, \quad (k_1 + ik_2)\ c_0 - (k_3 + 2\sigma)\ d_0 = 0 \end{array} \tag{43}$$

lead to only two eigenvalues:

$$\begin{array}{c} \sigma = \pm\dfrac{1}{2}\sqrt{k_1^2 + k_2^2 + k_3^2} = \pm\dfrac{1}{2}k\,, \\ b_0 = \dfrac{\pm k - k_3}{k_1 - ik_2}\ a_0 = \dfrac{k_1 + ik_2}{\pm k + k_3} a_0\,, \quad d_0 = \dfrac{\pm k - k_3}{k_1 - ik_2} c_0 = \dfrac{k_1 + ik_2}{\pm k + k_3}\ c_0\,. \end{array} \tag{44}$$

It should be emphasized that the systems (43) have nontrivial solution only for helicities $\sigma = \pm\frac{k}{2}$. We can easily check that formulas (. (28)) lead to expression for c_0, d_0 which agree with (44):

$$c_0 = \frac{E \mp k}{M}\ a_0\,, \quad d_0 = \frac{k_1 + ik_2}{\pm k + k_3}\left\{\frac{E \mp k}{M}\ a_0\right\} \quad \Longrightarrow \quad d_0 = \frac{k_1 + ik_2}{\pm k + k_3}\ c_0\,.$$

Now, let us examine equations for a_j, b_j:

$$\begin{array}{c} +(k_3 - 2\sigma)\ a_1 - 2ik_3\ a_2 + 2ik_2\ a_3 = -(k_1 - ik_2)\ b_1\,, \\ +2ik_3\ a_1 + (k_3 - 2\sigma)\ a_2 - 2ik_1\ a_3 = -(k_1 - ik_2)\ b_2\,, \\ -2ik_2\ a_1 + 2ik_1\ a_2 + (k_3 - 2\sigma)\ a_3 = -(k_1 - ik_2)\ b_3\,, \\ -(k_3 + 2\sigma)\ b_1 - 2ik_3\ b_2 + 2ik_2\ b_3 = -(k_1 + ik_2)\ a_1\,, \\ +2ik_3\ b_1 - (k_3 + 2\sigma)\ b_2 - 2ik_1\ b_3 = -(k_1 + ik_2)\ a_2\,, \\ -2ik_2\ b_1 + 2ik_1\ b_2 - (k_3 + 2\sigma)\ b_3 = -(k_1 + ik_2)\ a_3\,, \end{array} \tag{45}$$

and for c_j, d_j:

$$
\begin{aligned}
+(k_3-2\sigma)\,c_1 - 2ik_3\,c_2 + 2ik_2\,c_3 &= -(k_1-ik_2)\,d_1\,,\\
+2ik_3c_1 + (k_3-2\sigma)\,c_2 - 2ik_1c_3 &= -(k_1-ik_2)\,d_2\,,\\
-2ik_2c_1 + 2ik_1c_2 + (k_3-2\sigma)\,c_3 &= -(k_1-ik_2)\,d_3\,,\\
-(k_3+2\sigma)\,d_1 - 2ik_3\,d_2 + 2ik_2\,d_3 &= -(k_1+ik_2)\,c_1\,,\\
+2ik_3d_1 - (k_3+2\sigma)\,d_2 - 2ik_1d_3 &= -(k_1+ik_2)\,c_2\,,\\
-2ik_2d_1 + 2ik_1d_2 - (k_3+2\sigma)\,d_3 &= -(k_1+ik_2)\,c_3\,;
\end{aligned}
\tag{46}
$$

the system (46) coincides with (45), by this reason it ie enough to study only the system (45); let us present it differently

$$
\begin{aligned}
&\left|\begin{array}{ccc} +(k_3-2\sigma) & -2ik_3 & +2ik_2 \\ +2ik_3 & +(k_3-2\sigma) & -2ik_1 \\ -2ik_2 & +2ik_1 & +(k_3-2\sigma) \end{array}\right| \mathbf{a} = -(k_1-ik_2)\,\mathbf{b}\,,\\
&\left|\begin{array}{ccc} -(k_3+2\sigma) & -2ik_3 & +2ik_2 \\ +2ik_3 & -(k_3+2\sigma) & -2ik_1 \\ -2ik_2 & +2ik_1 & -(k_3+2\sigma) \end{array}\right| \mathbf{b} = -(k_1+ik_2)\,\mathbf{a}\,.
\end{aligned}
\tag{47}
$$

By exclusion method, we derive separate equations for variables $\mathbf{a}$ and $\mathbf{b}$:

$$
\left\{\left|\begin{array}{ccc} -(k_3+2\sigma) & -2ik_3 & +2ik_2 \\ +2ik_3 & -(k_3+2\sigma) & -2ik_1 \\ -2ik_2 & +2ik_1 & -(k_3+2\sigma) \end{array}\right|\left|\begin{array}{ccc} (k_3-2\sigma) & -2ik_3 & +2ik_2 \\ +2ik_3 & (k_3-2\sigma) & -2ik_1 \\ -2ik_2 & +2ik_1 & (k_3-2\sigma) \end{array}\right| - (k_1^2+k_2^2)\right\}\mathbf{a} = 0\,,
$$

$$
\left\{\left|\begin{array}{ccc} (k_3-2\sigma) & -2ik_3 & +2ik_2 \\ +2ik_3 & (k_3-2\sigma) & -2ik_1 \\ -2ik_2 & +2ik_1 & (k_3-2\sigma) \end{array}\right|\left|\begin{array}{ccc} -(k_3+2\sigma) & -2ik_3 & +2ik_2 \\ +2ik_3 & -(k_3+2\sigma) & -2ik_1 \\ -2ik_2 & +2ik_1 & -(k_3+2\sigma) \end{array}\right| - (k_1^2+k_2^2)\right\}\mathbf{b} = 0\,.
$$

We will solve equation for $\mathbf{a}$, and then find the vector $\mathbf{b}$ from (47).

First, let us consider the case of simple orientation of the plane wave:

$$
(0,0,k_3), \qquad \left|\begin{array}{ccc} 4\sigma^2+3k_3^2 & 8ik_3\sigma & 0 \\ -8ik_3\sigma & 4\sigma^2+3k_3^2 & 0 \\ 0 & 0 & 4\sigma^2-k_3^2 \end{array}\right|\left|\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\right| = 0\,,
\tag{48}
$$

whence we derive an equation with respect to σ:

$$
[(4\sigma^2+3k_3^2)^2 - 8^2k_3^2\sigma^2]\,(4\sigma^2-k_3^2) = 0\,,
$$

that is

$$
\sigma = -\frac{1}{2}k, -\frac{1}{2}k, \quad +\frac{1}{2}k, +\frac{1}{2}k, \quad -\frac{3}{2}k, \quad +\frac{3}{2}k\,;
\tag{49}
$$

the root $\pm k/2$ are doubly degenerate. Let us get corresponding solutions for $\mathbf{a}$:

$$
\left|\begin{array}{ccc} 4\sigma^2+3k_3^2 & 8ik_3\sigma & 0 \\ -8ik_3\sigma & 4\sigma^2+3k_3^2 & 0 \\ 0 & 0 & 4\sigma^2-k_3^2 \end{array}\right|\left|\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\right| = 0\,.
$$

At $\sigma = \pm\frac{1}{2}k_3$ we have the system

$$
\left|\begin{array}{ccc} 4k_3^2 & \pm 4ik_3^2 & 0 \\ \mp 4ik_3^2 & 4k_3^2 & 0 \\ 0 & 0 & 0 \end{array}\right|\left|\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\right| = 0,
$$

the rank of the matrix equal to 1, and solutions are

$$
\begin{aligned}
\sigma = -\frac{1}{2}k_3, \quad a_1 - ia_2 = 0\,, \quad a_3 \text{ (arbitrary)}; - \;\;;\\
\sigma = +\frac{1}{2}k_3, \quad a_1 + ia_2 = 0\,, \quad a_3 \text{ (arbitrary)},.
\end{aligned}
$$

For definiteness let $a_3 = \pm a_1$, correspondingly, at each σ we have two solutions:

$$\sigma = -\frac{1}{2}k_3, \quad \mathbf{a} = a_1(1, -i, \pm 1)\,; \qquad \sigma = +\frac{1}{2}k_3, \quad \mathbf{a} = a_1(1, +i, \pm 1)\,. \tag{50}$$

Now, let $\sigma = \pm 3/2$, then solutions are

$$\begin{array}{ll} \sigma = -\dfrac{3}{2}k_3, & \left|\begin{array}{ccc} 12k_3^2 & -12ik_3^2 & 0 \\ +12ik_3^2 & 12k_3^2 & 0 \\ 0 & 0 & 8k_3^2 \end{array}\right| \left|\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\right| = 0, \quad a_2 = -ia_1,\ a_3 = 0\,; \\ k = +\dfrac{3}{2}k_3, & \left|\begin{array}{ccc} 12k_3^2 & +12ik_3^2 & 0 \\ -12ik_3^2 & 12k_3^2 & 0 \\ 0 & 0 & 8k_3^2 \end{array}\right| \left|\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\right| = 0, \quad a_2 = +ia_1,\ a_3 = 0\,. \end{array} \tag{51}$$

Turn to the case of general orientation:

$$\left|\begin{array}{ccc} 4\sigma^2 - k^2 + 4(k_2^2 + k_3^2) & +8i\sigma k_3 - 4k_1k_2 & -8i\sigma k_2 - 4k_1k_3 \\ -8i\sigma k_3 - 4k_1k_2 & 4\sigma^2 - k^2 + 4(k_1^2 + k_3^2) & 8i\sigma k_1 - 4k_2k_3 \\ 8i\sigma k_2 - 4k_1k_3 & -8i\sigma k_1 - 4k_2k_3 & 4\sigma^2 - k^2 + 4(k_1^2 + k_2^2) \end{array}\right| \mathbf{a} = 0\,. \tag{52}$$

It is readily proved that from vanishing the main determinant follow yet known eigenvalues:

$$\sigma = -\frac{1}{2}k, -\frac{1}{2}k, \quad +\frac{1}{2}k, +\frac{1}{2}k, \quad -\frac{3}{2}k, +\frac{3}{2}k\,. \tag{53}$$

It is more convenient to use dimensionless quantities:

$$\frac{k_i}{k} = n_i\,, \quad n_in_i = 1\,, \quad \frac{\sigma}{k} \Longrightarrow \sigma\,, \qquad \sigma = -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, -\frac{3}{2}, +\frac{3}{2}\,,$$

then the above equation (52) reads

$$\left|\begin{array}{ccc} 4\sigma^2 - 1 + 4(n_2^2 + n_3^2) & +8i\sigma n_3 - 4n_1n_2 & -8i\sigma n_2 - 4n_1n_3 \\ -8i\sigma n_3 - 4n_1n_2 & 4\sigma^2 - 1 + 4(n_1^2 + n_3^2) & 8i\sigma n_1 - 4n_2n_3 \\ 8i\sigma n_2 - 4n_1n_3 & -8i\sigma n_1 - 4n_2n_3 & 4\sigma^2 - 1 + 4(n_1^2 + n_2^2) \end{array}\right| \mathbf{a} = 0\,. \tag{54}$$

Let $\sigma = \pm 1/2$:

$$\begin{array}{c} (n_2^2 + n_3^2)\ a_1 + (\pm in_3 - n_1n_2)\ a_2 - (\pm in_2 + n_1n_3)\ a_3 = 0\,, \\ -(\pm in_3 + n_1n_2)\ a_1 + (n_1^2 + n_3^2)\ a_2 + (\pm in_1 - n_2n_3)\ a_3 = 0\,, \\ (\pm in_2 - n_1n_3)\ a_1 - (\pm in_1 + n_2n_3)\ a_2 + (n_1^2 + n_2^2)\ a_3 = 0\,. \end{array} \tag{55}$$

The rank of the matrix equals to 1, only one equation is independent, let it be the third:

$$(\pm in_2 - n_1n_3)\ a_1 - (\pm in_1 + n_2n_3)\ a_2 + (1 - n_3^2)\ a_3 = 0\,. \tag{56}$$

This equation may have two independent solutions (which correlates with double multiplicity of the roots $\sigma = \pm 1/2$). The first and the most simple is $\mathbf{a}^{(1)} = (n_1, n_2, n_3)$; indeed

$$(\pm in_2 - n_1n_3)\ n_1 - (\pm in_1 + n_2n_3)\ n_2 + (1 - n_3^2)\ n_3 \equiv 0\,.$$

Taking in mind the structure of eq. (56) the second solution should have the form of vector product:

$$\mathbf{a}^{(2)} = \left|\begin{array}{ccc} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ n_1 & n_2 & n_3 \\ (\pm in_2 - n_1n_3) & -(\pm in_1 + n_2n_3) & (1 - n_3^2) \end{array}\right| = (\pm in_1n_3 + n_2)\,\mathbf{e}_1 + (\pm in_2n_3 - n_1)\,\mathbf{e}_2 \mp i(1 - n_3^2)\,\mathbf{e}_3\ ;$$

Thus, there exist two independent solutions of eq. (56) at $\sigma = \pm 1/2$:

$$\mathbf{a}^{(1)} = (n_1, n_2, n_3) = \mathbf{n}\,, \quad \mathbf{a}^{(2)} = (\ \pm in_1n_3 + n_2;\ \pm in_2n_3 - n_1;\ \mp i(1 - n_3^2)\ )\,; \tag{57}$$

recall that in the case $n_1 = 0, n_2 = 0$ these formula are not valid, instead result (50) should be used.

Now we study the case $\sigma = \pm 3/2$. The rank of the system (54) equals to 2, let us preserve two first equations:

$$
\begin{aligned}
(2 + n_2^2 + n_3^2)\, a_1 + (2i\sigma n_3 - n_1 n_2)\, a_2 &= (+2i\sigma n_2 + n_1 n_3)\, a_3 \,, \\
-(2i\sigma n_3 + n_1 n_2)\, a_1 + (2 + n_1^2 + n_3^2)\, a_2 &= (-2i\sigma n_1 + n_2 n_3)\, a_3 \,,
\end{aligned}
\tag{58}
$$

Determinant of this system equals to $6(1 - n_3^2)$, it vanishes at $n_3 = \pm 1$, this case is peculiar. In general case solution of the system (58) is

$\sigma = \pm 3/2$,

$$
\begin{aligned}
a_1 &= \frac{(3 - 4\sigma^2) n_1 n_3 + 4i\sigma n_2}{6 + (3 - 4\sigma^2) n_3^2}\, a_3 = \frac{-n_1 n_3 \pm i n_2}{1 - n_3^2}\, a_3 \,, \quad \text{let} \;\; a_3 = 1 - n_3^2 \,; \\
a_2 &= \frac{(3 - 4\sigma^2) n_2 n_3 - 4i\sigma n_1}{6 + (3 - 4\sigma^2) n_3^2}\, a_3 = \frac{-n_2 n_3 \mp i n_1}{1 - n_3^2}\, a_3 \,, \quad \text{let} \;\; a_3 = 1 - n_3^2 \,.
\end{aligned}
\tag{59}
$$

The choice of a_3 fixes normalization of this solution.

For each set $\{a_1, a_2, a_3\}_\sigma$ one can find related set $\{b_1, b_2, b_3\}_\sigma$ (see (47)):

$$
\left| \begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \right| = -\frac{1}{n_1 - i n_2} \left| \begin{array}{ccc} +(n_3 - 2\sigma) & -2i n_3 & +2i n_2 \\ +2i n_3 & +(n_3 - 2\sigma) & -2i n_1 \\ -2i n_2 & +2i n_1 & +(n_3 - 2\sigma) \end{array} \right| \left| \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right| ,
\tag{60}
$$

which may be presented in vector form

$$
\mathbf{b} = \frac{1}{n_1 - i n_2} \left[(2\sigma - n_3)\, \mathbf{a} - 2i\, \mathbf{n} \times \mathbf{a} \right] .
\tag{61}
$$

First, consider the case $\sigma = \pm 1/2$. For solutions of the type $\mathbf{a}^{(1)} = \mathbf{n}$ we derive

$$
\mathbf{b}^{(1)} = \frac{\pm 1 - n_3}{n_1 - i n_2}\, \mathbf{a}^{(1)} \,.
\tag{62}
$$

For solutions of the second type

$$
\mathbf{a}^{(2)} = (\pm i n_1 n_3 + n_2; \pm i n_2 n_3 - n_1; \mp i(1 - n_3^2)) \,, \quad \mathbf{b}^{(2)} = \frac{1}{n_1 - i n_2} [(\pm 1 - n_3)\, \mathbf{a}^{(2)} - 2i\, \mathbf{n} \times \mathbf{a}^{(2)}]
$$

allowing for identities

$$
\begin{aligned}
(\mathbf{n} \times \mathbf{a}^{(2)})_1 &= n_2 a_3^{(2)} - n_3 a_2^{(2)} = (\mp i) a_1^{(2)} \,, \\
(\mathbf{n} \times \mathbf{a}^{(2)})_2 &= n_3 a_1^{(2)} - n_1 a_3^{(2)} = (\mp i) a_2^{(2)} \,, \\
(\mathbf{n} \times \mathbf{a}^{(2)})_3 &= n_1 a_2^{(2)} - n_2 a_1^{(2)} = (\mp i) a_3^{(2)} \,.
\end{aligned}
\tag{63}
$$

we derive

$$
\mathbf{b}^{(2)} = \frac{1}{n_1 - i n_2} [(\pm 1 - n_3) - 2i\, (\mp i)]\, \mathbf{a}^{(2)} = \frac{(\mp 1 - n_3)}{n_1 - i n_2}\, \mathbf{a}^{(2)}.
\tag{64}
$$

It should be emphasized that in (62) and (64) multipliers are different.

Now consider the states with helicities $\sigma = \pm 3/2$:

$$
\begin{aligned}
a_1 = (-n_1 n_3 \pm i n_2) \,, \qquad a_2 &= (-n_2 n_3 \mp i n_1) \,, \qquad a_3 = 1 - n_3^2 \,; \\
b_1 = \frac{1}{n_1 - n_2} \{\, (\pm 3 - n_3) a_1 - 2i(n_2 a_3 - n_3 a_2) \,\} &= \frac{\pm 1 - n_3}{n_1 - i n_2}\, (-n_1 n_3 \pm i n_2) \,, \\
b_2 = \frac{1}{n_1 - n_2} \{\, (\pm 3 - n_3) a_2 - 2i(n_3 a_1 - n_1 a_3) \,\} &= \frac{\pm 1 - n_3}{n_1 - i n_2}\, (-n_2 n_3 \mp i n_1) \,, \\
b_2 = \frac{1}{n_1 - n_2} \{\, (\pm 3 - n_3) a_3 - 2i(n_1 a_2 - n_2 a_1) \,\} &= \frac{\pm 1 - n_3}{n_1 - i n_2}\, (1 - n_3^2) \,,
\end{aligned}
\tag{65}
$$

or shortly

$$\mathbf{b} = \frac{\pm 1 - n_3}{n_1 - in_2}\, \mathbf{a}\,. \tag{66}$$

Let us collect results together:

$$\begin{gathered}\sigma = \pm 1/2, \quad \{\, a_0, b_0, c_0, d_0, a_j^{(1)}, b_j^{(1)} \,\}, \quad \{\, a_0, b_0, c_0, d_0, a_j^{(2)}, b_j^{(2)} \,\},\\ b_0 = \frac{\pm 1 - n_3}{n_1 - in_2}\, a_0\,, \quad d_0 = \frac{\pm 1 - n_3}{n_1 - in_2}\, c_0\,,\end{gathered} \tag{67}$$

$$\mathbf{a}^{(1)} = \mathbf{n}\,, \quad \mathbf{b}^{(1)} = \frac{\pm 1 - n_3}{n_1 - in_2}\, \mathbf{a}^{(1)}, \quad \mathbf{c}^{(1)} \sim \mathbf{n}\,, \quad \mathbf{d}^{(1)} = \frac{\pm 1 - n_3}{n_1 - in_2}\, \mathbf{c}^{(1)}\,; \tag{68}$$

$$\begin{gathered}a_1^{(2)} = (\pm i n_1 n_3 + n_2),\; a_2^{(2)} = (\pm i n_2 n_3 - n_1),\; a_3^{(2)} = \mp i(1 - n_3^2)\,,\; \mathbf{b}^{(2)} = \frac{\mp 1 - n_3}{n_1 - in_2}\, \mathbf{a}^{(2)}\,,\\ c_1^{(2)} \sim (\pm i n_1 n_3 + n_2),\; c_2^{(2)} \sim (\pm i n_2 n_3 - n_1),\; c_3^{(2)} \sim \mp i(1 - n_3^2)\,,\; \mathbf{d}^{(2)} = \frac{\mp 1 - n_3}{n_1 - in_2}\, \mathbf{c}^{(2)}\,,\end{gathered} \tag{69}$$

$$\begin{gathered}\sigma = \pm 3/2, \qquad a_0 = 0\,, \quad b_0 = 0\,, \quad c_0 = 0\,, \quad d_0 = 0\,,\\ a_1 = (-n_1 n_3 \pm i n_2)\,, \quad a_2 = (-n_2 n_3 \mp i n_1)\,, \quad a_3 = 1 - n_3^2\,; \quad \mathbf{b} = \frac{\pm 1 - n_3}{n_1 - in_2}\, \mathbf{a}\,,\\ c_1 = (-n_1 n_3 \pm i n_2)\,, \quad c_2 = (-n_2 n_3 \mp i n_1)\,, \quad c_3 = 1 - n_3^2\,; \quad \mathbf{d} = \frac{\pm 1 - n_3}{n_1 - in_2}\, \mathbf{c}\,.\end{gathered} \tag{70}$$

V. HELICITY OPERATOR AND SOLUTIONS OF THE WAVE EQUATION

The above study of the wave equation give linear constraints

$$\begin{gathered}E a_0 + k_1 a_1 + k_2 a_2 + k_3 a_3 = 0\,, \quad E b_0 + k_1 b_1 + k_2 b_2 + k_3 b_3 = 0\,.\\ a_0 + a_3 = -(b_1 - i b_2)\,, \quad b_0 - b_3 = -(a_1 + i a_2)\,.\end{gathered} \tag{71}$$

and

$$c_l = \frac{E - k_3}{M}\, a_l - \frac{k_1 - i k_2}{M}\, b_l\,, \quad d_l = -\frac{k_1 + i k_2}{M}\, a_l + \frac{E + k_3}{M}\, b_l\,. \tag{72}$$

First, let us consider states with helicities $\sigma = \pm 3/2$:

$$\begin{gathered}a_0 = 0\,, \quad b_0 = 0\,, \quad c_0 = 0\,, \quad d_0 = 0\,,\\ a_1 = (-n_1 n_3 \pm i n_2)\,, \quad a_2 = (-n_2 n_3 \mp i n_1)\,, \quad a_3 = 1 - n_3^2\,; \quad \mathbf{b} = \frac{\pm 1 - n_3}{n_1 - in_2}\, \mathbf{a}\,,\\ c_1 \sim (-n_1 n_3 \pm i n_2)\,, \quad c_2 \sim (-n_2 n_3 \mp i n_1)\,, \quad c_3 \sim 1 - n_3^2\,; \quad \mathbf{d} = \frac{\pm 1 - n_3}{n_1 - in_2}\, \mathbf{c}\,;\end{gathered} \tag{73}$$

symbol remind that $\mathbf{c}$ is fixed up to arbitrary multiplier. Equations (71) take the form

$$\begin{gathered}n_1 a_1 + n_2 a_2 + n_3 a_3 = 0\,, \qquad n_1 b_1 + n_2 b_2 + n_3 b_3 = 0\,.\\ a_3 = -(b_1 - i b_2)\,, \qquad b_3 = (a_1 + i a_2)\,,\end{gathered} \tag{74}$$

Let us prove that relations (73) and (74) agree with each other. First, we take into account (73) in the first equation from (74):

$$n_1 a_1 + n_2 a_2 + n_3 a_3 = 0 \implies n_1(-n_1 n_3 \pm i n_2) + n_2(-n_2 n_3 \mp i n_1) + n_3(n_1^2 + n_2^2) \implies 0 = 0\,.$$

We do not need re-check second relation from (74). Then, let us check relation $a_3 = -(b_1 - ib_2)$:

$$-(b_1 - ib_2) = -\frac{\pm 1 - n_3}{n_1 - in_2}\{(-n_1 n_3 \pm in_2) - i(-n_2 n_3 \mp in_1)\} =$$
$$= -\frac{\pm 1 - n_3}{n_1 - in_2}\{n_1(-n_3 \mp 1) - in_2(-n_3 \mp 1)\} = -(\pm 1 - n_3)(\mp 1 - n_3) = 1 - n_3^2 = a_3\,. \tag{75}$$

Now, let us check relation $b_3 = (a_1 + ia_2)$:

$$(a_1 + ia_2) = (-n_1 n_3 \pm in_2) + i(-n_2 n_3 \mp in_1) = -n_1 n_3 \pm in_2 - in_2 n_3 \pm n_1 = (n_1 + in_2)(\pm 1 - n_3) \equiv b_3\,.$$

Taking in mind (74), we find c_j:

$$c_j = \frac{E - k_3}{M} a_j - \frac{k_1 - ik_2}{M}\,\frac{\pm k - k_3}{k_1 - ik_2}\,a_j = \frac{E \mp k}{M}\,a_j\,. \tag{76}$$

let us find d_j:

$$d_j = -\frac{k_1 + ik_2}{M}\,a_j + \frac{E + k_3}{M}\,\frac{\pm k - k_3}{k_1 - ik_2}\,\frac{k_1 + ik_2}{k_1 + ik_2}\,a_j =$$

$$= -\frac{k_1 + ik_2}{M}\,a_j \pm \frac{E + k_3}{M}\,\frac{k \mp k_3}{(k - k_3)(k + k_3)}\,(k_1 + ik_2)\,a_j = \frac{k_1 + ik_2}{k_3 \pm k}\left\{\frac{E \mp k}{M}\,a_j\right\}\,,$$

that is

$$d_j = \frac{k_1 + ik_2}{k_3 \pm k}\,c_j = \frac{\pm k - k_3}{k_1 - ik_2}\,c_j\,, \tag{77}$$

this coincides with the (72). Thus, states with helicities $\sigma = \pm 3/2$ actually are exact solutions of the wave equation.

Now we turn to the case of $\sigma = \pm 1/2$. Because we have double generation, first let us consider reach of them separately. States of the first type are given by (see (44))

$$b_0^{(1)} = \frac{\pm 1 - n_3}{n_1 - in_2}\,a_0^{(1)} = \frac{n_1 + in_2}{\pm 1 + n_3}\,a_0^{(1)}\,, \quad d_0^{(1)} = \frac{\pm 1 - n_3}{n_1 - in_2}\,c_0^{(1)} = \frac{n_1 + in_2}{\pm 1 + n_3}\,c_0^{(1)}\,,$$
$$\mathbf{a}^{(1)} = \mathbf{n}\,, \qquad \mathbf{b}^{(1)} = \frac{\pm 1 - n_3}{n_1 - in_2}\,\mathbf{a}^{(1)}\,, \quad \mathbf{c}^{(1)} \sim \mathbf{a}^{(1)}\,, \qquad \mathbf{d}^{(1)} = \frac{\pm 1 - n_3}{n_1 - in_2}\,\mathbf{c}^{(1)}\,. \tag{78}$$

The symbol $\sim$ may be specified with the help of linear relations

$$\mathbf{c}^{(1)} = \frac{E - k_3}{M}\,\mathbf{a}^{(1)} - \frac{k_1 - ik_2}{M}\,\mathbf{b}^{(1)} = \frac{E \mp k}{M}\,\mathbf{a}^{(1)}\,,$$
$$\mathbf{d}^{(1)} = -\frac{k_1 + ik_2}{M}\,\mathbf{a}^{(1)} + \frac{E + k_3}{M}\,\mathbf{b}^{(1)} = \frac{k_1 + ik_2}{k_3 \pm k}\left\{\frac{E \mp k}{M}\mathbf{a}^{(1)}\right\} = \frac{\pm 1 - n_3}{n_1 - in_2}\,\mathbf{c}^{(1)}\,. \tag{79}$$

We can easily see that these states do not provide us with solutions of the wave equation. Indeed, we have (see (71))

$$Ea_0^{(1)} + k_1 a_1^{(1)} + k_2 a_2^{(1)} + k_3 a_3^{(1)} = 0 \quad \Longrightarrow \quad Ea_0^{(1)} + k = 0\,, \quad a_0^{(1)} = -\frac{k}{E}\,. \tag{80}$$

Equation $Eb_0^{(1)} + k_1 b_1^{(1)} + k_2 b_2^{(1)} + k_3 b_3^{(1)} = 0$ gives the same result. Let us check two remaining equations (see (71)):

$$a_0^{(1)} + a_3^{(1)} = -(b_1^{(1)} - ib_2^{(1)}) \implies -\frac{k}{E} + n_3 = -\frac{\pm 1 - n_3}{n_1 - in_2}(n_1 - in_2) = \pm 1 + n_3 \implies \frac{k}{E} = \mp 1\,;$$

$$b_0^{(1)} - b_3^{(1)} = -(a_1^{(1)} + ia_2^{(1)}) \implies \frac{\pm 1 - n_3}{n_1 - in_2}(-\frac{k}{E} - n_3) = -(n_1 + in_2) \implies$$

$$(\pm 1 - n_3)(\frac{k}{E} + n_3) = (1 - n_3)(1 + n_3) \quad \Longrightarrow \quad \frac{k}{E} + n_3 = \pm 1 + n_3, \quad \frac{k}{E} = \pm 1\,;$$

these two identity cannot be valid, because $E=\sqrt{k^2+M^2}$.

Now consider states with $\sigma=\pm 1/2$ of second type. First, note relations

$$b_0^{(2)} = \frac{\pm 1 - n_3}{n_1 - in_2}\, a_0^{(2)}\,, \quad d_0^{(2)} = \frac{\pm 1 - n_3}{n_1 - in_2}\, c_0^{(2)}\,, \quad c_0^{(2)} = \frac{E \mp k}{M}\, a_0^{(2)}\,. \tag{81}$$

Now consider vectors $\mathbf{a}^{(2)}, \mathbf{b}^{(2)}, \mathbf{c}^{(2)}, \mathbf{d}^{(2)}$:

$$a_1^{(2)} = \pm i n_1 n_3 + n_2\,, \quad a_2^{(2)} = \pm i n_2 n_3 - n_1\,, \quad a_3^{(2)} = \mp i(1-n_3^2)\,, \quad \mathbf{b}^{(2)} = \frac{\mp 1 - n_3}{n_1 - in_2}\, \mathbf{a}^{(2)}\,,$$

$$c_1^{(2)} \sim (\pm i n_1 n_3 + n_2), \quad c_2^{(2)} \sim (\pm i n_2 n_3 - n_1), \quad c_3^{(2)} \sim \mp i(1-n_3^2)\ )\,, \quad \mathbf{d}^{(2)} = \frac{\mp 1 - n_3}{n_1 - in_2}\, \mathbf{c}^{(2)}\,,$$

Symbol $\sim$ may be specified with help of linear restrictions (see (72)):

$$\begin{aligned} \mathbf{c}^{(2)} &= \frac{E-k_3}{M}\, \mathbf{a}^{(2)} - \frac{k_1 - ik_2}{M}\, \mathbf{b}^{(2)} = \frac{E \pm k}{M}\, \mathbf{a}^{(2)}\,, \\ \mathbf{d}^{(2)} &= -\frac{k_1 + ik_2}{M}\, \mathbf{a}^{(2)} + \frac{E+k_3}{M}\, \mathbf{b}^{(2)} = \frac{\mp 1 - n_3}{n_1 - in_2}\, \mathbf{c}^{(2)}\,. \end{aligned} \tag{82}$$

Let us introduce notations:

$$\Gamma = \frac{\pm 1 - n_3}{n_1 - in_2}\,, \quad R = \frac{\mp 1 - n_3}{n_1 - in_2}\,, \tag{83}$$

then the above formulas read shorter

$$b_0^{(2)} = \Gamma\, a_0^{(2)}\,, \quad d_0^{(2)} = \Gamma\, c_0^{(a)}\,, \quad \mathbf{b}^{(2)} = R\, \mathbf{a}^{(2)}\,, \quad \mathbf{d}^{(2)} = R\, \mathbf{c}^{(2)}\,. \tag{84}$$

Now we may check the first consequence of the wave equation (see (71))

$$E a_0^{(2)} + k_1 a_1^{(2)} + k_2 a_2^{(2)} + k_3\,, a_3^{(2)} = 0 \tag{85}$$

it yields

$$E a_0^{(2)} + k\{\ n_1(\pm i n_1 n_3 + n_2) + n_2(\pm i n_2 n_3 - n_1) + n_3(\mp i)(n_1^2 + n_2^2)\ \} = 0 \quad \Longrightarrow \quad a_0^{(2)} = 0\,.$$

Similarly, we derive

$$E b_0^{(2)} + k_1 b_1^{(2)} + k_2^{(2)} b_2 + k_3 b_3^{(2)} = 0 \quad \Longrightarrow \quad b_0^{(2)} = 0\,.$$

Thus, we obtain

$$a_0^{(2)} = 0, \quad b_0^{(2)} = 0, \quad c_0^{(2)} = 0, \quad d_0^{(2)} = 0\,. \tag{86}$$

let us check two remaining restrictions (see (71))

$$a_3^{(2)} = -(b_1^{(2)} - i b_2^{(2)})\,, \quad a_3^{(2)} = \mp i(1 - n_3^2)\,,$$

$$-(b_1^{(2)} - i b_2^{(2)}) = -R\ [\pm i n_1 n_3 + n_2 - i(\pm i n_2 n_3 - n_1)] =$$

$$= \pm i(1 \pm n_3)(1 \pm n_3)\ \frac{1 - n_3^2}{(1-n_3)(1+n_3)} = \pm i(1 - n_3^2)\ \frac{1 \pm n_3}{1 \mp n_3} = \frac{R}{\Gamma} a_3^{(2)} \neq a_3^{(2)}\,,$$

we conclude that needed restriction does not hold. Similarly, we derive

$$b_3^{(2)} = a_1^{(2)} + i a_2^{(2)}\,,$$

$$b_3^{(2)} = \frac{\mp 1 - n_3}{n_1 - in_2}(\mp i)(1-n_3^2) = \frac{\mp 1 - n_3}{n_1 - in_2}(\mp i)(n_1 - in_2)(n_1+in_2) = i(n_1+in_2)(1 \pm n_3)$$

$$a_1^{(2)} + ia_2^{(2)} = \pm i n_1 n_3 + n_2 + i(\pm i n_2 n_3 - n_1) =$$

$$= -i(n_1+in_2)(1\mp n_3)\cdot\frac{(1\pm n_3)}{(1\pm n_3)} = -i(n_1+in_2)(1\pm n_3)\cdot\frac{(1\mp n_3)}{(1\pm n_3)} = \frac{\Gamma}{R}\, b_3^{(2)} \neq b_3^{(2)}\,,$$

again the needed restriction does not hold. Therefore, each separate helicity state, of type (1) and (2), with $\sigma = \pm 1/2$ does not provide us with solution of the wave equation.

Let us search solution of the wave equations as a linear combination of these states:

$$\begin{gathered} a_0 = Fa_0^{(1)} + Ga_0^{(2)}, \quad \mathbf{a} = F\mathbf{a}^{(1)} + G\mathbf{a}^{(2)}\,, \\ b_0 = F\Gamma a_0^{(1)} + G\Gamma a_0^{(2)}\,, \quad \mathbf{b} = F\Gamma\mathbf{a}^{(1)} + GR\mathbf{a}^{(2)}\,; \end{gathered} \tag{87}$$

remind that normalization of vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{b}^{(1)}, \mathbf{b}^{(2)}$ is fixed, bur parameters $a_0^{(1)}, a_0^{(2)}$ are arbitrary.

Two first restrictions from (71) give

$$Ea_0 + \mathbf{ka} = 0 \quad \Longrightarrow \quad E(Fa_0^{(1)} + Ga_0^{(2)}) + F\,k + G\cdot 0 = 0\,,$$

$$Eb_0 + \mathbf{kb} = 0 \quad \Longrightarrow \quad E\Gamma(Fa_0^{(1)} + Ga_0^{(2)}) + F\Gamma k + G\cdot 0 = 0\,,$$

that is

$$Fa_0^{(1)} + Ga_0^{(2)} + F\,\frac{k}{E} = 0 \;\Longrightarrow\; F(a_0^{(1)} + \frac{k}{E}) + Ga_0^{(2)} = 0\,. \tag{88}$$

taking in mind (80) and (85)

$$a_0^{(1)} = -\frac{k}{E}, \quad a_0^{(2)} = 0$$

we conclude that these two equations do not impose any restrictions on parameters F, G.

Now consider two remaining equations from (71):

$$a_0 + a_3 = -(b_1 - ib_2)\,, \quad b_0 - b_3 = -(a_1 + ia_2)\,.$$

they are equivalent to

$$(Fa_0^{(1)} + Ga_0^{(2)}) + (Fa_3^{(1)} + Ga_3^{(2)}) = -\Big\{(F\Gamma a_1^{(1)} + GRa_1^{(2)}) - i(F\Gamma a_2^{(1)} + GRa_2^{(2)}))\Big\}\,,$$

$$(F\Gamma a_0^{(1)} + G\Gamma a_0^{(2)}) - (F\Gamma a_3^{(1)} + GRa_3^{(2)}) = -\Big\{(Fa_1^{(1)} + Ga_1^{(2)}) + i(Fa_2^{(1)} + Ga_2^{(2)})\Big\}\,,$$

which after re-grouping the terms lead to

$$\begin{gathered} F\Big[(a_0^{(1)} + a_3^{(1)}) + \Gamma(a_1^{(1)} - ia_2^{(1)})\Big] + G\Big[(a_0^{(2)} + a_3^{(2)}) + R(a_1^{(2)} - ia_2^{(2)})\Big] = 0\,, \\ F\Big[\Gamma(a_0^{(1)} - a_3^{(1)}) + (a_1^{(1)} + ia_2^{(1)})\Big] + G\Big[(\Gamma a_0^{(2)} - Ra_3^{(2)}) + (a_1^{(2)} + ia_2^{(2)})\Big] = 0\,. \end{gathered} \tag{89}$$

Whence, taking into account $a_0^{(1)} = -k/E$ and $a_0^{(2)} = 0$ we derive following two equations:

$$\begin{gathered} F\Big[-k/E + a_3^{(1)} + \Gamma(a_1^{(1)} - ia_2^{(1)})\Big] + G\Big[a_3^{(2)} + R(a_1^{(2)} - ia_2^{(2)})\Big] = 0\,, \\ F\Big[-\Gamma k/E - \Gamma a_3^{(1)} + (a_1^{(1)} + ia_2^{(1)})\Big] + G\Big[-Ra_3^{(2)} + (a_1^{(2)} + ia_2^{(2)})\Big] = 0\,. \end{gathered} \tag{90}$$

Now we should use the formulas

$$a_1^{(1)} = n_1 , \quad a_2^{(1)} = n_2 , \quad a_3^{(1)} = n_3 , \quad \Gamma = \frac{\pm 1 - n_3}{n_1 - in_2} \, , \; R = \frac{\mp 1 - n_3}{n_1 - in_2} \, ,$$
$$a_1^{(2)} = \pm i n_1 n_3 + n_2 , \quad a_2^{(2)} = \pm i n_2 n_3 - n_1 , \quad a_3^{(2)} = \mp i (1 - n_3^2) \, .$$

First, we find

$$a_3^{(1)} + \Gamma (a_1^{(1)} - i a_2^{(1)}) = n_1 + \frac{\pm 1 - n_3}{n_1 - in_2} (n_1 - in_2) = \pm 1 \, ,$$
$$a_3^{(2)} + R (a_1^{(2)} - i a_2^{(2)}) = \mp i [(1 - n_3 (1 + n_3) + (1 \pm n_3)(1 \pm n_3)] = \mp 2i (1 \pm n_3) \, ;$$

therefore the first equation in (90) take the form

$$(- \frac{k}{E} \pm 1) \; F \mp 2i (1 \pm n_3)) \; G = 0 \tag{91}$$

Similarly, we get

$$-\Gamma a_3^{(1)} + (a_1^{(1)} + i a_2^{(1)}) = - \frac{\pm 1 - n_3}{n_1 - in_2} n_3 + (n_1 + in_2) \frac{n_1 - in_2}{n_1 - in_2} =$$
$$= \frac{1}{n_1 - in_2} [(\mp 1 + n_3) n_3 + (1 - n_3)(1 + n_3)] = \frac{1 \mp n_3}{n_1 - in_2} \, ,$$
$$-R a_3^{(2)} + (a_1^{(2)} + i a_2^{(2)}) = \pm i (\mp 1 - n_3)(n_1 + in_2) \pm i (n_1 + in_2)(n_3 \mp 1) =$$
$$= \pm i (n_1 + in_2) [\mp 1 - n_3 + n_3 \mp 1] = -2i (n_1 + in_2) \; ;$$

therefore, the second equation in (90) takes the form

$$F \left[\mp \frac{1 \mp n_3}{n_1 - in_2} \frac{k}{E} + \frac{1 \mp n_3}{n_1 - in_2} \right] + G \left[-2i (n_1 + in_2) \right] = 0 \, ,$$

or

$$\mp \frac{1 \mp n_3}{n_1 - in_2} (\frac{k}{E} \mp 1) \; F - 2i (n_1 + in_2) \; G = 0 \, ,$$

that is

$$(\frac{k}{E} \mp 1) \; F \pm 2i (1 \pm n_3) \; G = 0 \, ; \tag{92}$$

note that eq. (92) coincides with the first one (91).

Thus, we have found coefficients of the needed linear combination of two type of states with helicity $\sigma = \pm 1/2$. These coefficient are fixed up to some total arbitrary multiplier which relate to normalization of solutions./

VI. THE PLANE WAVE SOLUTIONS

In this section we turn to the massless spin 3/2 particle. The system (20) may be written in matrix form (note changing the notation, $\tilde{\Psi}^l(x) = \Phi^l(x)$):

$$\left| \begin{array}{cccc} 0 & (\gamma^2 \partial_3 - \gamma^3 \partial_2) & (\gamma^3 \partial_1 - \gamma^1 \partial_3) & (\gamma^1 \partial_2 - \gamma^2 \partial_1) \\ (\gamma^2 \partial_3 - \gamma^3 \partial_2) & 0 & -(\gamma^3 \partial_0 + \gamma^0 \partial_3) & (\gamma^2 \partial_0 + \gamma^0 \partial_2) \\ (\gamma^3 \partial_1 - \gamma^1 \partial_3) & (\gamma^3 \partial_0 + \gamma^0 \partial_3) & 0 & -(\gamma^1 \partial_0 + \gamma^0 \partial_1) \\ (\gamma^1 \partial_2 - \gamma^2 \partial_1) & -(\gamma^2 \partial_0 + \gamma^0 \partial_2) & (\gamma^1 \partial_0 + \gamma^0 \partial_1) & 0 \end{array} \right| \left| \begin{array}{c} \Phi^0 \\ \Phi^1 \\ \Phi^2 \\ \Phi^3 \end{array} \right| = 0 \, . \tag{93}$$

We will search solutions in the form of plane waves

$$\Phi^l(x) = e^{i k_a x^a} A^l , \quad k_a x^a = k_0 x^0 + k_j x^j , \quad k_0 = -\epsilon \, ,$$

where $A^l = (A^0, A^1, A^2, A^3)$ stand for 4-columns, then eq. (93) takes the form

$$\left| \begin{array}{cccc} 0 & (\gamma^2 k_3 - \gamma^3 k_2) & (\gamma^3 k_1 - \gamma^1 k_3) & (\gamma^1 k_2 - \gamma^2 k_1) \\ (\gamma^2 k_3 - \gamma^3 k_2) & 0 & -(\gamma^3 k_0 + \gamma^0 k_3) & (\gamma^2 k_0 + \gamma^0 k_2) \\ (\gamma^3 k_1 - \gamma^1 k_3) & (\gamma^3 k_0 + \gamma^0 k_3) & 0 & -(\gamma^1 k_0 + \gamma^0 k_1) \\ (\gamma^1 k_2 - \gamma^2 k_1) & -(\gamma^2 k_0 + \gamma^0 k_2) & (\gamma^1 k_0 + \gamma^0 k_1) & 0 \end{array} \right| \left| \begin{array}{c} A^0 \\ A^1 \\ A^2 \\ A^3 \end{array} \right| = 0\,. \tag{94}$$

It is readily seen that gradient-type solutions $\Phi_l^{(0)}(x) = \partial_l e^{ikx}\varphi$ satisfy this equation identically

$$\left| \begin{array}{cccc} 0 & (\gamma^2 k_3 - \gamma^3 k_2) & (\gamma^3 k_1 - \gamma^1 k_3) & (\gamma^1 k_2 - \gamma^2 k_1) \\ (\gamma^2 k_3 - \gamma^3 k_2) & 0 & -(\gamma^3 k_0 + \gamma^0 k_3) & (\gamma^2 k_0 + \gamma^0 k_2) \\ (\gamma^3 k_1 - \gamma^1 k_3) & (\gamma^3 k_0 + \gamma^0 k_3) & 0 & -(\gamma^1 k_0 + \gamma^0 k_1) \\ (\gamma^1 k_2 - \gamma^2 k_1) & -(\gamma^2 k_0 + \gamma^0 k_2) & (\gamma^1 k_0 + \gamma^0 k_1) & 0 \end{array} \right| \left| \begin{array}{c} k^0\varphi \\ -k_1\varphi \\ -k_2\varphi \\ -k_3\varphi \end{array} \right| = 0\,. \tag{95}$$

It is convenient to present the system (94) as four equations:

$$\begin{array}{r} (\gamma^2 k_3 - \gamma^3 k_2) A^1 + (\gamma^3 k_1 - \gamma^1 k_3) A^2 + (\gamma^1 k_2 - \gamma^2 k_1) A^3 = 0\,, \\ (\gamma^3 k_0 + \gamma^0 k_3) A^2 - (\gamma^2 k_0 + \gamma^0 k_2) A^3 = (\gamma^2 k_3 - \gamma^3 k_2) A^0\,, \\ -(\gamma^3 k_0 + \gamma^0 k_3) A^1 + (\gamma^1 k_0 + \gamma^0 k_1) A^3 = (\gamma^3 k_1 - \gamma^1 k_3) A^0\,, \\ (\gamma^2 k_0 + \gamma^0 k_2) A^1 - (\gamma^1 k_0 + \gamma^0 k_1) A^2 = (\gamma^1 k_2 - \gamma^2 k_1) A^0\,. \end{array} \tag{96}$$

From the first equation, allowing for identity $(\gamma^1 k_2 - \gamma^2 k_1)(\gamma^1 k_2 - \gamma^2 k_1) = -k_2^2 - k_1^2$, follows expression for A^3:

$$\begin{array}{l} (k_1^2 + k_2^2) A^3 = (\gamma^1\gamma^2 k_2 k_3 + k_1 k_2 \gamma^2\gamma^3 + k_3 k_1 + \gamma^3\gamma^1 k_2^2)\, A^1 + \\ \qquad +(\gamma^1\gamma^3 k_2 k_1 + k_1 k_3 \gamma^2\gamma^1 + k_3 k_2 + \gamma^3\gamma^2 k_1^2)\, A^2. \end{array} \tag{97}$$

Similarly, from fourth equation, we get

$$\begin{array}{l} -(k_1^2 + k_2^2) A^0 = (\gamma^1\gamma^2 k_0 k_2 + k_1 k_2 \gamma^0\gamma^2 + k_1 k_0 + \gamma^1\gamma^0 k_2^2)\, A^1 + \\ \qquad +(\gamma^2\gamma^1 k_0 k_1 + k_2 k_1 \gamma^0\gamma^1 + k_2 k_0 + \gamma^2\gamma^0 k_1^2)\, A^2. \end{array} \tag{98}$$

For brevity, let us apply notations

$$k_1 = a, \quad k_2 = b, \quad k_3 = c\,, \quad k_0 = d\,, \quad d^2 - a^2 - b^2 - c^2 = 0\,.$$

Then, expressions for A^3, A^0 read

$$\begin{array}{r} (a^2 + b^2) A^3 = (bc\gamma^1\gamma^2 + ab\gamma^2\gamma^3 + ac + b^2\gamma^3\gamma^1) A^1 + (ab\gamma^1\gamma^3 + ac\gamma^2\gamma^1 + bc + a^2\gamma^3\gamma^2) A^2, \\ -(a^2 + b^2) A^0 = (db\gamma^1\gamma^2 + ab\gamma^0\gamma^2 + da + b^2\gamma^1\gamma^0) A^1 + (da\gamma^2\gamma^1 + ab\gamma^0\gamma^1 + db + a^2\gamma^2\gamma^0) A^2. \end{array} \tag{99}$$

With the help of (99), one can exclude the variables A^0, A^3 from -nd and 3-rd equations in (96). So we find two equations with respect to A^1 and A^2:

$$\begin{array}{l} ab\{d\,\gamma^1\gamma^2\gamma^3 + c\,\gamma^0\gamma^1\gamma^2 + a\,\gamma^0\gamma^2\gamma^3 + b\,\gamma^0\gamma^3\gamma^1\} A^2 = \\ = b^2\{d\,\gamma^1\gamma^2\gamma^3 + c\,\gamma^0\gamma^1\gamma^2 + a\,\gamma^0\gamma^2\gamma^3 + b\,\gamma^0\gamma^3\gamma^1\} A^1\,, \end{array} \tag{100}$$

$$\begin{array}{l} ab\{d\,\gamma^1\gamma^2\gamma^3 + c\,\gamma^0\gamma^1\gamma^2 + a\,\gamma^0\gamma^2\gamma^3 + b\,\gamma^0\gamma^3\gamma^1\} A^1 = \\ = a^2\{d\,\gamma^1\gamma^2\gamma^3 + c\,\gamma^0\gamma^1\gamma^2 + a\,\gamma^0\gamma^2\gamma^3 + b\,\gamma^0\gamma^3\gamma^1\} A^2. \end{array} \tag{101}$$

With notation

$$K = d\,\gamma^1\gamma^2\gamma^3 + c\,\gamma^0\gamma^1\gamma^2 + a\,\gamma^0\gamma^2\gamma^3 + b\,\gamma^0\gamma^3\gamma^1\,, \tag{102}$$

eqs. (100)–(101) may be re-written as

$$aK\, A^2 = bKA^1,\; bK\, A^1 = aKA^2 \quad \Longrightarrow \quad K(aA^2 - bA^1) = 0\,,\; K(bA^1 - aA^2) = 0\,.$$

In fact, here we have only one equation (further the notation $k_a = (d, a, b, c)$ is not applied)

$$K(k_1 A^2 - k_2 A^1) = 0\,, \tag{103}$$

where

$$K = k_0 \gamma^1\gamma^2\gamma^3 + k_1\gamma^0\gamma^2\gamma^3 + k_2\gamma^0\gamma^3\gamma^1 + k_3\gamma^0\gamma^1\gamma^2\,. \tag{104}$$

taking into account the definition

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \left|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right|, \quad (\gamma^5)^2 = I, \quad \gamma^5\gamma^a = -\gamma^a\gamma^5\,, \tag{105}$$

and identities

$$\begin{array}{c} i\gamma^5\gamma^0 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = \gamma^1\gamma^2\gamma^3, \quad i\gamma^5\gamma^1 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 = \gamma^0\gamma^2\gamma^3, \\ i\gamma^5\gamma^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = \gamma^0\gamma^3\gamma^1, \quad i\gamma^5\gamma^3 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = \gamma^0\gamma^1\gamma^2 \end{array} \tag{106}$$

we obtain different representation for matrix K:

$$K = i\gamma^5(k_0\gamma^0 + k_1\gamma^1 + k_2\gamma^2 + k_3\gamma^3)\,. \tag{107}$$

Therefore, eq. (103) is written as (the multiplier $i\gamma^5$ is omitted):

$$(k_0\gamma^0 + k_1\gamma^1 + k_2\gamma^2 + k_3\gamma^3)A = 0\,, \quad A = (k_1 A^2 - k_2 A^1)\,. \tag{108}$$

Equation (108) may be presented as (let $A = (a, b, c, d)$

$$\left|\begin{array}{cccc} 0 & 0 & k^0 - k_3 & -k_1 + ik_2 \\ 0 & 0 & -k_1 - ik_2 & k^0 + k_3 \\ k^0 + k_3 & k_1 - ik_2 & 0 & 0 \\ k_1 + ik_2 & k^0 - k_3 & 0 & 0 \end{array}\right| \left|\begin{array}{c} a \\ b \\ c \\ d \end{array}\right| = 0\,,$$

whence follows two linear systems with respect to d, d and a, b (let $k_0 = -\epsilon, \epsilon > 0$):

$$(\epsilon + k_3)c + (k_1 - ik_2)d = 0\,, \quad (k_1 + ik_2)c + (\epsilon - k_3)d = 0\,;$$

$$(\epsilon - k_3)a - (k_1 - ik_2)b = 0\,, \quad -(k_1 + ik_2)a + (\epsilon + k_3)b = 0\,.$$

Their solutions are

$$b = \frac{\epsilon - k_3}{k_1 - ik_2}a = \frac{k_1 + ik_2}{\epsilon + k_3}a\,, \qquad d = -\frac{\epsilon + k_3}{k_1 - ik_2}c = -\frac{k_1 + ik_2}{\epsilon - k_3}c\,. \tag{109}$$

Temporary we will use the shortening form for solutions (109):

$$b = (\epsilon - k_3)a\,, \quad d = -(\epsilon + k_3)c\,; \tag{110}$$

in fact we male change the notations

$$\frac{\epsilon - k_3}{k_1 - ik_2} \Longrightarrow \epsilon - k_3, \qquad \frac{\epsilon + k_3}{k_1 - ik_2} \Longrightarrow \epsilon + k_3\,. \tag{111}$$

However, in the very end these changes (111) should have been remembered.

Therefore, general solution may be presented as

$$A = -k_2 A^1 + k_1 A^2 = a \left|\begin{array}{c} 1 \\ (\epsilon - k_3) \\ 0 \\ 0 \end{array}\right| + c \left|\begin{array}{c} 0 \\ 0 \\ 1 \\ -(\epsilon + k_3) \end{array}\right|. \tag{112}$$

Because parameters a and c are independent, in (112) we have two linearly independent solutions. Let us fix solutions $A_{(1)}$ and $A_{(2)}$ in the following way:

$$(1), \quad a = k_1, c = -k_2, \quad A_{(1)} = -k_2 A^1_{(1)} + k_1 A^2_{(1)} = -k_2 \begin{vmatrix} 0 \\ 0 \\ 1 \\ -(\epsilon + k_3) \end{vmatrix} + k_1 \begin{vmatrix} 1 \\ (\epsilon - k_3) \\ 0 \\ 0 \end{vmatrix} ; \tag{113}$$

$$(2), \quad a = -k_2, c = k_1, \quad A_{(2)} = -k_2 A^1_{(2)} + k_1 A^2_{(2)} = -k_2 \begin{vmatrix} 1 \\ (\epsilon - k_3) \\ 0 \\ 0 \end{vmatrix} + k_1 \begin{vmatrix} 0 \\ 0 \\ 1 \\ -(\epsilon + k_3) \end{vmatrix} , \tag{114}$$

or differently

$$(1) \qquad A^1_{(1)} = \begin{vmatrix} 0 \\ 0 \\ 1 \\ -(\epsilon + k_3) \end{vmatrix} , \quad A^2_{(1)} = \begin{vmatrix} 1 \\ (\epsilon - k_3) \\ 0 \\ 0 \end{vmatrix} ; \tag{115}$$

$$(2) \qquad A^1_{(2)} = \begin{vmatrix} 1 \\ (\epsilon - k_3) \\ 0 \\ 0 \end{vmatrix} , \quad A^2_{(2)} = \begin{vmatrix} 0 \\ 0 \\ 1 \\ -(\epsilon + k_3) \end{vmatrix} . \tag{116}$$

below we will take into account four identities

$$k_1 A^1_{(1)} + k_2 A^2_{(1)} = \begin{vmatrix} k_2 \\ k_2(\epsilon - k_3) \\ k_1 \\ -k_1(\epsilon + k_3) \end{vmatrix} , \quad k_1 A^1_{(2)} + k_2 A^2_{(2)} = \begin{vmatrix} k_1 \\ k_1(\epsilon - k_3) \\ k_2 \\ -k_2(\epsilon + k_3) \end{vmatrix} , \tag{117}$$

$$-k_2 A^1_{(1)} + k_1 A^2_{(1)} = \begin{vmatrix} k_1 \\ k_1(\epsilon - k_3) \\ -k_2 \\ +k_2(\epsilon + k_3) \end{vmatrix} , \quad -k_2 A^1_{(2)} + k_1 A^2_{(2)} = \begin{vmatrix} -k_2 \\ -k_2(\epsilon - k_3) \\ k_1 \\ -k_1(\epsilon + k_3) \end{vmatrix} . \tag{118}$$

Concomitant components $A^0_{(1)}, A^3_{(1)}$ and $A^0_{(2)}, A^3_{(2)}$ can ve calculated according to (97) and (98); let us write down them em differently:

$$(k_1^2 + k_2^2) A^3 = k_3 (k_1 A^1 + k_2 A^2) - (k_1 \gamma^2 \gamma^3 + k_2 \gamma^3 \gamma^1 + k_3 \gamma^1 \gamma^2)(-k_2 A^1 + k_1 A^2) , \tag{119}$$

$$-(k_1^2 + k_2^2) A^0 = k_0 (k_1 A^1 + k_2 A^2) + (\gamma^0 \gamma^1 k_2 - \gamma^0 \gamma^2 k_1 - k_0 \gamma^1 \gamma^2)(-k_2 A^1 + k_1 A^2) , \tag{120}$$

whence we get

$$(k_1^2 + k_2^2) A^3 = k_3 (k_1 A^1 + k_2 A^2) + i \begin{vmatrix} k_3 & k_1 - ik_2 & 0 & 0 \\ k_1 + ik_2 & -k_3 & 0 & 0 \\ 0 & 0 & k_3 & k_1 - ik_2 \\ 0 & 0 & k_1 + ik_2 & -k_3 \end{vmatrix} (-k_2 A^1 + k_1 A^2) ,$$

$$-(k_1^2 + k_2^2) A^0 = k_0 (k_1 A^1 + k_2 A^2) + i \begin{vmatrix} k_0 & k_1 - ik_2 & 0 & 0 \\ -k_1 - ik_2 & -k_0 & 0 & 0 \\ 0 & 0 & k_0 & -k_1 + ik_2 \\ 0 & 0 & k_1 + ik_2 & -k_0 \end{vmatrix} (-k_2 A^1 + k_1 A^2) .$$

Further we can obtain expressions for $A^0_{(1)}, A^3_{(1)}$ and $A^0_{(2)}, A^3_{(2)}$:

$$A^3_{(1)} = \begin{vmatrix} (k_1 + ik_2)^{-1} [ik_3 + ik_1(\epsilon - k_3)] \\ (k_1 - ik_2)^{-1} [ik_1 - ik_3(\epsilon - k_3)] \\ (k_1 + ik_2)^{-1} [k_3 + ik_2(\epsilon + k_3)] \\ (k_1 - ik_2)^{-1} [-ik_2 - k_3(\epsilon + k_3)] \end{vmatrix} , A^0_{(1)} = \begin{vmatrix} (k_1 + ik_2)^{-1} [-ik_0 - ik_1(\epsilon - k_3)] \\ (k_1 - ik_2)^{-1} [+ik_1 + ik_0(\epsilon - k_3)] \\ (k_1 + ik_2)^{-1} [-k_0 + ik_2(\epsilon + k_3)] \\ (k_1 - ik_2)^{-1} [+ik_2 + k_0(\epsilon + k_3)] \end{vmatrix} ; \tag{121}$$

$$A^3_{(2)} = \begin{vmatrix} (k_1+ik_2)^{-1}[+k_3 - ik_2(\epsilon-k_3)] \\ (k_1-ik_2)^{-1}[-ik_2 + k_3(\epsilon-k_3)] \\ (k_1+ik_2)^{-1}[+ik_3 - ik_1(\epsilon+k_3)] \\ (k_1-ik_2)^{-1}[+ik_1 + ik_3(\epsilon+k_3)] \end{vmatrix}, A^0_{(2)} = \begin{vmatrix} (k_1+ik_2)^{-1}[-k_0 + ik_2(\epsilon-k_3)] \\ (k_1-ik_2)^{-1}[-ik_2 - k_0(\epsilon-k_3)] \\ (k_1+ik_2)^{-1}[-ik_0 - ik_1(\epsilon+k_3)] \\ (k_1-ik_2)^{-1}[-ik_1 - ik_0(\epsilon+k_3)] \end{vmatrix}. \tag{122}$$

VII. RELATION TO INITIAL BASIS

Recall formulas relating two bases:

$$\Psi_l = (\delta^k_l - \gamma_l\gamma^k)\tilde{\Psi}_k, \quad \Psi_l(x) = e^{ikx}B_l\,, \quad \tilde{\Psi}_l(x) = e^{ikx}A_l, \quad l = 0,1,2,3\,. \tag{123}$$

In component form they read

$$\begin{aligned} B^0 &= A^0 - \gamma^0(\gamma^0A^0 - \gamma^1A^1 - \gamma^2A^2 - \gamma^3A^3)\,, \\ B^1 &= A^1 - \gamma^1(\gamma^0A^0 - \gamma^1A^1 - \gamma^2A^2 - \gamma^3A^3)\,, \\ B^2 &= A^2 - \gamma^2(\gamma^0A^0 - \gamma^1A^1 - \gamma^2A^2 - \gamma^3A^3)\,, \\ B^3 &= A^3 - \gamma^3(\gamma^0A^0 - \gamma^1A^1 - \gamma^2A^2 - \gamma^3A^3)\,. \end{aligned} \tag{124}$$

Let us find the blocks

$$(\gamma^0A^0 - \gamma^1A^1 - \gamma^2A^2 - \gamma^3A^3)_{(1)} = \begin{vmatrix} (k_1+ik_2)^{-1}[-k_0 + 2ik_2(\epsilon+k_3) + k_3] - (\epsilon+k_3) \\ (k_1-ik_2)^{-1}[2ik_2 + (k_0+k_3)(\epsilon+k_3)] + 1 \\ (k_1+ik_2)^{-1}i[-k_0 - 2k_1(\epsilon-k_3) - k_3] + i(\epsilon-k_3) \\ (k_1-ik_2)^{-1}i[2k_1 + (k_0-k_3)(\epsilon-k_3)] - i \end{vmatrix}, \tag{125}$$

$$(\gamma^0A^0 - \gamma^1A^1 - \gamma^2A^2_{(2)} - \gamma^3A^3)_{(2)} = \begin{vmatrix} (k_1+ik_2)^{-1}i[-k_0 - 2k_1(\epsilon+k_3) + k_3] + i(\epsilon+k_3) \\ (k_1-ik_2)^{-1}i[-2k_1 - (k_0+k_3)(\epsilon+k_3)] + i \\ (k_1+ik_2)^{-1}[-k_0 + 2ik_2(\epsilon-k_3) - k_3] - (\epsilon-k_3) \\ (k_1-ik_2)^{-1}[-2ik_2 - (k_0-k_3)(\epsilon-k_3)] - 1 \end{vmatrix}. \tag{126}$$

Now, (125) and (126) are to be taken into account in (124)/ The last may be split in two groups, after needed calculations we get

$$B^1_{(1)} = \begin{vmatrix} (k_1-ik_2)^{-1}i[+2k_1 + (k_0-k_3)(\epsilon-k_3)] - i \\ (k_1+ik_2)^{-1}i[-k_0 - 2k_1(\epsilon-k_3) - k_3] + i(\epsilon-k_3) \\ (k_1-ik_2)^{-1}[-2ik_2 - (k_0+k_3)(\epsilon+k_3)] \\ (k_1+ik_2)^{-1}[+k_0 - 2ik_2(\epsilon+k_3) - k_3] \end{vmatrix} = \begin{vmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{vmatrix}_{(1)},$$

$$B^1_{(2)} = \begin{vmatrix} (k_1-ik_2)^{-1}[-2ik_2 - (k_0-k_3)(\epsilon-k_3)] \\ (k_1+ik_2)^{-1}[-k_0 + 2ik_2(\epsilon-k_3) - k_3] \\ (k_1-ik_2)^{-1}i[+2k_1 + (k_0+k_3)(\epsilon+k_3)] - i \\ (k_1+ik_2)^{-1}i[+k_0 + 2k_1(\epsilon+k_3) - k_3] - i(\epsilon+k_3) \end{vmatrix} = \begin{vmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{vmatrix}_{(2)},$$

$$B^2_{(1)} = \begin{vmatrix} (k_1-ik_2)^{-1}[+2k_1 + (k_0-k_3)(\epsilon-k_3)] \\ (k_1+ik_2)^{-1}[+k_0 + 2k_1(\epsilon-k_3) + k_3] \\ (k_1-ik_2)^{-1}[-2k_2 + i(k_0+k_3)(\epsilon+k_3)] + i \\ (k_1+ik_2)^{-1}[+ik_0 + 2k_2(\epsilon+k_3) - ik_3] + i(\epsilon+k_3) \end{vmatrix} = \begin{vmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{vmatrix}_{(1)},$$

$$B^2_{(2)} = \begin{vmatrix} (k_1-ik_2)^{-1}[-2k_2+i(k_0-k_3)(\epsilon-k_3)]+i \\ (k_1+ik_2)^{-1}[-ik_0-2k_2(\epsilon-k_3)-ik_3]-i(\epsilon-k_3) \\ (k_1-ik_2)^{-1}[+2k_1+(k_0+k_3)(\epsilon+k_3)] \\ (k_1+ik_2)^{-1}[-k_0-2k_1(\epsilon+k_3)+k_3] \end{vmatrix} = \begin{vmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{vmatrix}_{(2)} ,$$

$$B^3_{(1)} = \begin{vmatrix} (k_1+ik_2)^{-1}[-ik_1(\epsilon-k_3)-ik_3]+i(\epsilon-k_3) \\ (k_1-ik_2)^{-1}[-ik_1-ik_0(\epsilon-k_3)]+i \\ (k_1+ik_2)^{-1}[-ik_2(\epsilon+k_3)-k_3]+(\epsilon+k_3) \\ (k_1-ik_2)^{-1}[+ik_2+k_0(\epsilon+k_3)]+1 \end{vmatrix} = \begin{vmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{vmatrix}_{(1)} ,$$

$$B^3_{(2)} = \begin{vmatrix} (k_1+ik_2)^{-1}[+ik_2(\epsilon-k_3)-k_3]-(\epsilon-k_3) \\ (k_1-ik_2)^{-1}[+ik_2+k_0(\epsilon-k_3)]+1 \\ (k_1+ik_2)^{-1}[+ik_0+ik_1(\epsilon+k_3)]-i(\epsilon+k_3) \\ (k_1-ik_2)^{-1}[-ik_1-ik_0(\epsilon+k_3)]+i \end{vmatrix} = \begin{vmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{vmatrix}_{(2)} ,$$

$$B^0_{(1)} = \begin{vmatrix} (k_1+ik_2)^{-1}[+ik_1(\epsilon-k_3)+ik_3]-i(\epsilon-k_3) \\ (k_1-ik_2)^{-1}[-ik_1+ik_3(\epsilon-k_3)]+i \\ (k_1+ik_2)^{-1}[-ik_2(\epsilon+k_3)-k_3]+(\epsilon+k_3) \\ (k_1-ik_2)^{-1}[-ik_2-k_3(\epsilon+k_3)]-1 \end{vmatrix} = \begin{vmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{vmatrix}_{(1)} ,$$

$$B^0_{(2)} = \begin{vmatrix} (k_1+ik_2)^{-1}[-ik_2(\epsilon-k_3)+k_3]+(\epsilon-k_3) \\ (k_1-ik_2)^{-1}[+ik_2-k_3(\epsilon-k_3)]+1 \\ (k_1+ik_2)^{-1}[+ik_1(\epsilon+k_3)-ik_3]-i(\epsilon+k_3) \\ (k_1-ik_2)^{-1}[+ik_1+ik_3(\epsilon+k_3)]-i \end{vmatrix} = \begin{vmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{vmatrix}_{(2)} . \tag{127}$$

These formulas (127) may be simplified, by taking in mind the following identities and notations

$$k_0 = -\epsilon,\ (\epsilon+k_3)(\epsilon-k_3) = (k_1-ik_2)(k_1+ik_2)\,,\ \epsilon = k,\ d\frac{k_j}{\epsilon} = n_j,\ n_j n_j = 1\,.$$

In this way we derive

$$\begin{aligned} a^{(1)}_1 = 0, \quad a^{(1)}_2 = 1, \quad a^{(1)}_3 = -\frac{n_2}{1+n_3} - \frac{in_3}{n_1+in_2}, \quad a^{(1)}_0 = \frac{n_2}{1+n_3} + \frac{in_3}{n_1+in_2}\,, \\ a^{(2)}_1 = 1, \quad a^{(2)}_2 = 0, \qquad a^{(2)}_3 = -\frac{n_1}{1+n_3} - \frac{n_3}{n_1+in_2}, \quad a^{(2)}_0 = \frac{n_1}{1+n_3} + \frac{n_3}{n_1+in_2}\,, \end{aligned} \tag{128}$$

$$\begin{aligned} b^{(1)}_1 = 0, \quad b^{(1)}_2 = \frac{n_1+in_2}{1+n_3}\,, \quad b^{(1)}_3 = \frac{n_2n_3+in_1}{(n_1-in_2)(1+n_3)}\,, \quad b^{(1)}_0 = \frac{n_2+in_1n_3}{(n_1-in_2)(1+n_3)}\,, \\ b^{(2)}_1 = \frac{n_1+in_2}{1+n_3}, \quad b^{(2)}_2 = 0, \quad b^{(2)}_3 = \frac{n_1n_3-in_2}{(n_1-in_2)(1+n_3)}\,, \quad b^{(2)}_0 = \frac{n_1-in_2n_3}{(n_1-in_2)(1+n_3)}\,, \end{aligned} \tag{129}$$

$$\begin{aligned} c^{(1)}_1 = 1, \quad c^{(1)}_2 = 0, \quad c^{(1)}_3 = \frac{n_1}{1-n_3} - \frac{n_3}{n_1+in_2}, \quad c^{(1)}_0 = \frac{n_1}{1-n_3} - \frac{n_3}{n_1+in_2}\,, \\ c^{(2)}_1 = 0, \quad c^{(2)}_2 = 1, \quad c^{(2)}_3 = \frac{n_2}{1-n_3} - \frac{i}{n_1+in_2}, \quad c^{(2)}_0 = \frac{n_2}{1-n_3} - \frac{in_3}{n_1+in_2}\,, \end{aligned} \tag{130}$$

$$d_1^{(1)} = -\frac{n_1 + in_2}{1-n_3}, \quad d_2^{(1)} = 0, \quad d_3^{(1)} = \frac{-n_1 n_3 - in_2}{(1-n_3)(n_1 - in_2)}, \quad d_0^{(1)} = \frac{-n_1 - in_2 n_3}{(1-n_3)(n_1 - in_2)},$$
$$d_1^{(2)} = 0, \quad d_2^{(2)} = -\frac{n_1 + in_2}{1-n_3}, \quad d_3^{(2)} = \frac{-n_2 n_3 + in_1}{(1-n_3)(n_1 - in_2)}, \quad d_0^{(2)} = \frac{-n_2 + in_1 n_3}{(1-n_3)(n_1 - in_2)}. \tag{131}$$

VIII. HELICITY OPERATOR

In [...], eigenvectors of helicity operator for spin 3/2 particles we constructed (the case of a massive particle was studied). Let us recall the used notations and results.

Vector-bispinor wave function $\Psi_l(x)$ is written as a matrix (A is a bispinor index, (l) is vector one)

$$\Psi_{A(l)}(x) = \left| \begin{array}{cccc} \Psi_{1(0)}(x) & \Psi_{1(1)}(x) & \Psi_{1(2)}(x) & \Psi_{1(3)}(x) \\ \Psi_{2(0)}(x) & \Psi_{2(1)}(x) & \Psi_{2(2)}(x) & \Psi_{2(3)}(x) \\ \Psi_{3(0)}(x) & \Psi_{3(1)}(x) & \Psi_{3(2)}(x) & \Psi_{3(3)}(x) \\ \Psi_{4(0)}(x) & \Psi_{4(1)}(x) & \Psi_{4(2)}(x) & \Psi_{4(3)}(x) \end{array} \right| = e^{ikx} \left| \begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{array} \right| = e^{ikx}\{A_0, A_1, A_2, A_3\}. \tag{132}$$

There are possible 4 eigenvalues σ for helicity operator (the values $\pm$ are doubly degenerate)

$$\sigma = -\frac{1}{2}k, -\frac{1}{2}k, \quad +\frac{1}{2}k, +\frac{1}{2}k, \quad -\frac{3}{2}k, \quad +\frac{3}{2}k \quad k = \sqrt{k_1^2 + k_2^2 + k_2^2}; \tag{133}$$

dimensionless quantities are more convenient:

$$\frac{k_i}{k} = n_i, \quad n_i n_i = 1, \quad \frac{\sigma}{k} \Longrightarrow \sigma, \qquad \sigma = -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, -\frac{3}{2}, +\frac{3}{2}. \tag{134}$$

For each value from $\sigma = \pm 1/2$ thee exist two eigenstates (they are marked by (I) and (II)):

$$\Psi_0^I = \left| \begin{array}{c} 1 \\ \frac{\pm 1 - n_3}{n_1 - n_2} \\ 1 \\ \frac{\pm 1 - n_3}{n_1 - n_2} \end{array} \right|, \quad \Psi_1^I = \left| \begin{array}{c} n_1 \\ n_1 \frac{\pm 1 - n_3}{n_1 - n_2} \\ n_1 \\ n_1 \frac{\pm 1 - n_3}{n_1 - n_2} \end{array} \right|, \quad \Psi_2^I = \left| \begin{array}{c} n_2 \\ n_2 \frac{\pm 1 - n_3}{n_1 - n_2} \\ n_2 \\ n_3 \frac{\pm 1 - n_3}{n_1 - n_2} \end{array} \right|, \quad \Psi_3^I = \left| \begin{array}{c} n_3 \\ n_3 \frac{\pm 1 - n_3}{n_1 - n_2} \\ n_3 \\ n_3 \frac{\pm 1 - n_3}{n_1 - n_2} \end{array} \right|, \tag{135}$$

$$\Psi_0^{II} = \left| \begin{array}{c} 1 \\ \frac{\pm 1 - n_3}{n_1 - n_2} \\ 1 \\ \lambda_2' \frac{\pm 1 - n_3}{n_1 - n_2} \end{array} \right|, \quad \Psi_1^{II} = \left| \begin{array}{c} (\pm i n_1 n_3 + n_2) \\ (\pm i n_1 n_3 + n_2) \frac{\mp 1 - n_3}{n_1 - n_2} \\ (\pm i n_1 n_3 + n_2) \\ (\pm i n_1 n_3 + n_2) \frac{\mp 1 - n_3}{n_1 - n_2} \end{array} \right|,$$
$$\Psi_2^{II} = \left| \begin{array}{c} (\pm i n_2 n_3 - n_1) \\ (\pm i n_2 n_3 - n_1) \frac{\mp 1 - n_3}{n_1 - n_2} \\ (\pm i n_2 n_3 - n_1) \\ (\pm i n_2 n_3 - n_1) \frac{\mp 1 - n_3}{n_1 - n_2} \end{array} \right|, \quad \Psi_3^{II} = \left| \begin{array}{c} [\mp i(1 - n_3^2)] \\ [\mp i(1 - n_3^2)] \frac{\mp 1 - n_3}{n_1 - n_2} \\ [\mp i(1 - n_3^2)] \\ [\mp i(1 - n_3^2)] \frac{\mp 1 - n_3}{n_1 - n_2} \end{array} \right|, \tag{136}$$

Eigenstates for helicities $\sigma = \pm 3/2$ are given by the formulas

$$\Psi_0^{III} = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right|, \quad \Psi_1^{III} = \left| \begin{array}{c} (-n_1 n_3 \pm i n_2) \\ (-n_1 n_3 \pm i n_2) \frac{\pm 1 - n_3}{n_1 - n_2} \\ (-n_1 n_3 \pm i n_2) \\ (-n_1 n_3 \pm i n_2) \frac{\pm 1 - n_3}{n_1 - n_2}, \end{array} \right|,$$
$$\Psi_2^{III} = \left| \begin{array}{c} (-n_2 n_3 \mp i n_1) \\ (-n_2 n_3 \mp i n_1) \frac{\pm 1 - n_3}{n_1 - n_2} \\ (-n_2 n_3 \mp i n_1) \\ (-n_2 n_3 \mp i n_1) \frac{\mp 1 - n_3}{n_1 - n_2}, \end{array} \right|, \quad \Psi_3^{III} = \left| \begin{array}{c} (1 - n_3^2) \\ (1 - n_3^2) \frac{\pm 1 - n_3}{n_1 - n_2} \\ (1 - n_3^2) \\ (1 - n_3^2) \frac{\mp 1 - n_3}{n_1 - n_2}, \end{array} \right|, \tag{137}$$

We are to relate helicity states (135)–(137) with solutions for massless particle:

$$B_{(1)}^0 = \left| \begin{array}{c} a_0 \\ b_0 \\ c_0 \\ d_0 \end{array} \right|_{(1)}, B_{(1)}^1 = \left| \begin{array}{c} a_1 \\ b_1 \\ c_1 \\ d_1 \end{array} \right|_{(1)}, B_{(1)}^2 = \left| \begin{array}{c} a_2 \\ b_2 \\ c_2 \\ d_2 \end{array} \right|_{(1),}, B_{(1)}^3 = \left| \begin{array}{c} a_3 \\ b_3 \\ c_3 \\ d_3 \end{array} \right|_{(1)}, \tag{138}$$

$$B^0_{(2)} = \left| \begin{array}{c} a_0 \\ b_0 \\ c_0 \\ d_0 \end{array} \right|_{(2)} , B^1_{(2)} = \left| \begin{array}{c} a_1 \\ b_1 \\ c_1 \\ d_1 \end{array} \right|_{(2)} , B^2_{(2)} = \left| \begin{array}{c} a_2 \\ b_2 \\ c_2 \\ d_2 \end{array} \right|_{(2),} , B^3_{(2)} = \left| \begin{array}{c} a_3 \\ b_3 \\ c_3 \\ d_3 \end{array} \right|_{(2)} . \qquad (139)$$

For helicities $\sigma = \pm 3/2$, the quantities a_0, b_0, c_0, d_0 equal to zero, however the quantities a_0, b_0, c_0, d_0 in massless solutions do not, therefore they cannot be constructed only in terms of helicity solutions with $\sigma = \pm 3/2$.

Besides, it should be emphasized that in describing helicity eigenstates the structure $\{a_0, a_1, a_2, a_3\}$ is the main one, and it determines all remaining quantities by means of linear relations. By this reason, it suffices to connect the quantities $\{a_0, a_1, a_2, a_3\}_{(1,2)}$ in (139) with relevant variables in (135)–(137).

Let us write down two sets of parameters for helicity states

$$\begin{gathered} a_0^{I+} = 1, \quad a_1^{I+} = n_1, \quad a_2^{I+} = n_2, \quad a_3^{I+} = n_3, \\ a_0^{I-} = 1, \quad a_1^{I-} = n_1, \quad a_2^{I-} = n_2, \quad a_3^{I-} = n_3, \\ a_0^{II+} = 1, \quad a_1^{II+} = in_1n_3 + n_2, \quad a_2^{II+} = in_2n_3 - n_1, \quad a_3^{II+} = -i(1-n_3^2), \\ a_0^{II-} = 1, \quad a_1^{II-} = -in_1n_3 + n_2, \quad a_2^{II-} = -in_2n_3 - n_1, \quad a_3^{II-} = +i(1-n_3^2), \\ a_0^{III+} = 0, \quad a_1^{III+} = -n_1n_3 + in_2, \quad a_2^{III+} = -n_2n_3 - in_1, \quad a_3^{III+} = (1-n_3^2), \\ a_0^{III-} = 0, \quad a_1^{III-} = -n_1n_3 - in_2, \quad a_2^{III-} = -n_2n_3 + in_1, \quad a_3^{III-} = (1-n_3^2), \end{gathered} \qquad (140)$$

and for massless solutions

$$\begin{array}{ll} (1), & a_0^{(1)} = \dfrac{n_2}{1+n_3} + \dfrac{in_3}{n_1+in_2}, \quad a_1^{(1)} = 0, \quad a_2^{(1)} = 1, \quad a_3^{(1)} = -\dfrac{n_2}{1+n_3} - \dfrac{in_3}{n_1+in_2}, \\ (2), & a_0^{(2)} = \dfrac{n_1}{1+n_3} + \dfrac{n_3}{n_1+in_2}, \quad a_1^{(2)} = 1, \quad a_2^{(2)} = 0, \quad a_3^{(2)} = -\dfrac{n_1}{1+n_3} - \dfrac{n_3}{n_1+in_2}. \end{array} \qquad (141)$$

Instead of variables in (141), it is more convenient to use their linear combinations:

$$a_l^+ = a_l^{(2)} + ia_l^{(1)}, \quad a_l^- = a_l^{(2)} - ia_l^{(1)}, \qquad (142)$$

they seem to be simpler

$$\begin{gathered} a_0^+ = \frac{n_1+in_2}{1+n_3}, \quad a_1^+ = 1, \quad a_2^+ = +i, \quad a_3^+ = -\frac{n_1+in_2}{1+n_3}, \\ a_0^- = \frac{1+n_3}{n_1+in_2}, \quad a_1^- = 1, \quad a_2^- = -i, \quad a_3^- = -\frac{1+n_3}{n_1+in_2}. \end{gathered} \qquad (143)$$

We search for linear expansions of massless solutions (143) ny helicity states (the index l takes on the values $0, 1, 2, 3$)

$$\begin{gathered} a_l^+ = \alpha a_l^{I+} + \alpha' a_l^{I-} + \beta a_l^{II+} + \beta' a_l^{II-} + \gamma a_l^{III+} + \gamma' a_l^{III-}, \\ a_l^- = \alpha a_l^{I+} + \alpha' a_l^{I-} + \beta a_l^{II+} + \beta' a_l^{II-} + \gamma a_l^{III+} + \gamma' a_l^{III-}; \end{gathered} \qquad (144)$$

in both formulas two first terms define only one parameter, $(\alpha + \alpha')$. Each equation provides us with four linear relations:

a_l^+-solution,

$$\begin{gathered} a_0^+ = \frac{n_1+in_2}{1+n_3} = (\alpha+\alpha') + (\beta+\beta') + \gamma \cdot 0 + \gamma' \cdot 0, \\ a_1^+ = 1 = (\alpha+\alpha')n_1 + \beta(in_1n_3+n_2) + \beta'(-in_1n_3+n_2) + \gamma(-n_1n_3+in_2) + \gamma'(-n_1n_3-in_2), \\ a_2^+ = i = (\alpha+\alpha')n_2 + \beta(in_2n_3-n_1) + \beta'(-in_2n_3-n_1) + \gamma(-n_2n_3-in_1) + \gamma'(-n_2n_3+in_1), \\ a_3^+ = -\frac{n_1+in_2}{1+n_3} = (\alpha+\alpha')n_3 + \beta(-i)(1-n_3^2) + \beta' i(1-n_3^2) + \gamma(1-n_3^2) + \gamma'(1-n_3^2); \end{gathered} \qquad (145)$$

a_l^--solution,

$$
\begin{aligned}
a_0^- &= \frac{1+n_3}{n_1+in_2} = (\alpha+\alpha') + (\beta+\beta') + \gamma\cdot 0 + \gamma'\cdot 0/ \\
a_1^- &= 1 = (\alpha+\alpha')n_1 + \beta(in_1n_3+n_2) + \beta'(-in_1n_3+n_2) + \gamma(-n_1n_3+in_2) + \gamma'(-n_1n_3-in_2), \\
a_2^- &= -i = (\alpha+\alpha')n_2 + \beta(in_2n_3-n_1) + \beta'(-in_2n_3-n_1) + \gamma(-n_2n_3-in_1) + \gamma'(-n_2n_3+in_1), \\
a_{30}^- &= -\frac{1+n_3}{n_1+in_2} = (\alpha+\alpha')n_3 + \beta(-i)(1-n_3^2) + \beta' i(1-n_3^2) + \gamma(1-n_3^2) + \gamma'(1-n_3^2)\,.
\end{aligned}
\tag{146}
$$

Whence after regrouping the terms we obtain (let $\alpha+\alpha'=\sigma$)

a_l^+-solution,

$$
\begin{aligned}
a_0^+ &= \frac{n_1+in_2}{1+n_3} = \sigma + (\beta+\beta'), \\
a_1^+ &= 1 = \sigma n_1 + (\beta+\beta')n_2 + i(\beta-\beta')n_1n_3 - (\gamma+\gamma')n_1n_3 + i(\gamma-\gamma')n_2 \\
a_2^+ &= i = \sigma n_2 - (\beta+\beta')n_1 + i(\beta-\beta')n_2n_3 - (\gamma+\gamma')n_2n_3 - i(\gamma-\gamma')n_1, \\
a_3^+ &= -\frac{n_1+in_2}{1+n_3} = \sigma n_3 - i(\beta-\beta')(1-n_3^2) + (\gamma+\gamma')(1-n_3^2)\,;
\end{aligned}
\tag{147}
$$

a_l^--solution,

$$
\begin{aligned}
a_0^- &= \frac{1+n_3}{n_1+in_2} = \sigma + (\beta+\beta'), \\
a_1^- &= 1 = \sigma n_1 + (\beta+\beta')n_2 + i(\beta-\beta')n_1n_3 - (\gamma+\gamma')n_1n_3 + i(\gamma-\gamma')n_2 \\
a_2^- &= -i = \sigma n_2 - (\beta+\beta')n_1 + i(\beta-\beta')n_2n_3 - (\gamma+\gamma')n_2n_3 - i(\gamma-\gamma')n_1, \\
a_3^- &= -\frac{1+n_3}{n_1+in_2} = \sigma n_3 - i(\beta-\beta')(1-n_3^2) + (\gamma+\gamma')(1-n_3^2)\,.
\end{aligned}
\tag{148}
$$

With new notations

$$
\sigma = x_0, \quad \beta+\beta' = x_1, \quad i(\beta-\beta') = x_2, \quad \gamma+\gamma' = x_3, \quad i(\gamma-\gamma') = x_4, \tag{149}
$$

the above systems read (it should be notified that coordinates x_2, x_3 enter these system only as the combination $y = x_2 - x_3$):

a_l^+-solution,

$$
\left|\begin{array}{cccc} 1 & 1 & 0 & 0 \\ n_1 & n_2 & n_1n_3 & n_2 \\ n_2 & -n_1 & n_2n_3 & -n_1 \\ n_3 & 0 & -(1-n_3^2) & 0 \end{array}\right| \left|\begin{array}{c} x_0 \\ x_1 \\ y \\ x_4 \end{array}\right| = \left|\begin{array}{c} \frac{n_1+in_2}{1+n_3} \\ 1 \\ +i \\ -\frac{n_1+in_2}{1+n_3} \end{array}\right| = \left|\begin{array}{c} a_0^+ \\ a_1^+ \\ a_2^+ \\ a_3^+ \end{array}\right|; \tag{150}
$$

a_l^--solution,

$$
\left|\begin{array}{cccc} 1 & 1 & 0 & 0 \\ n_1 & n_2 & n_1n_3 & n_2 \\ n_2 & -n_1 & n_2n_3 & -n_1 \\ n_3 & 0 & -(1-n_3^2) & 0 \end{array}\right| \left|\begin{array}{c} x_0 \\ x_1 \\ y \\ x_4 \end{array}\right| = \left|\begin{array}{c} \frac{n_1-in_2}{1-n_3} \\ 1 \\ -i \\ -\frac{n_1-in_2}{1=n_3} \end{array}\right| = \left|\begin{array}{c} a_0^- \\ a_1^- \\ a_2^- \\ a_3^- \end{array}\right|. \tag{151}
$$

We have two similar systems with the same main determinant, $n_1^2+n_1^2$, they may be readily solved – see below. From eqs. (150) and (151) it follows that coefficients in expansion of any massless solution by helicity ones

$$
a_l = x_0 a_l^I + \beta a_l^{II+} + \beta' a_l^{II-} + \gamma a_l^{III+} + \gamma' a_l^{III-} \tag{152}
$$

may be presented as linear combinations of four variables $x_0; x_1, y, x_4$; three last of them, x_0, x_1, y, x_4, determine constraints on $\beta, \beta', \gamma, \gamma'$ (see (149)):

$$
\begin{array}{lcl}
\beta+\beta' = x_1\,, & & \beta+\beta' = x_1\,, \\
i(\beta-\beta') - (\gamma+\gamma') = y\,, & \Longrightarrow & \beta-\beta' = -iy - i(\gamma+\gamma')\,, \\
i(\gamma-\gamma') = x_4\,, & & i(\gamma+\gamma') = x_4 + 2i\gamma'\,,
\end{array}
$$

which is equivalent to

$$\beta = \frac{1}{2}(x_1 - iy - x_4) - i\gamma' \,, \;\; \beta = \frac{1}{2}(x_1 + iy + x_4) + i\gamma' \,, \;\; \gamma = -ix_4 + \gamma' \,. \tag{153}$$

Taking into account (153) in expansion (152), we may decompose this expansion into the sum of two terms, (the seconde one is proportional to an arbitrary parameter γ'):

$$a_l = \left\{ x_0 a_l^I + \frac{1}{2}(x_1 - iy - x_4) a_l^{II+} \frac{1}{2}(x_1 + iy + x_4) a_l^{II-} - ix_4 a_l^{III+} \right\} - \\ -i\gamma' \left\{ a_l^{II+} - a_l^{II-} + i a_l^{III+} + i\gamma' a_l^{III-} \right\} . \tag{154}$$

Applying explicit form of helicity solutions, we prove that this additional term vanish identically.

Therefore, the final result reduces to expansion

$$a_l = \left\{ x_0 a_l^I + \frac{1}{2}(x_1 - iy - x_4) a_l^{II+} \frac{1}{2}(x_1 + iy + x_4) a_l^{II-} - ix_4 a_l^{III+} \right\} . \tag{155}$$

In order to find explicit expression (155) in the case of special massless states, a_l^+ and a_l^-, we need solutions of linear system (150) and (151):

a_l^+-solution,

$$x_0 = \frac{1 - n_3}{n_1 - in_2}, \quad x_1 = 0, \quad y = \frac{1}{n_1 - n_2}, \quad x_4 = -\frac{i}{n_1 - in_2} \,. \tag{156}$$

a_l^--solution,

$$x_0 = (1 + n_3) \frac{1 - 2n_3}{n_1 + in_2} \,, \quad x_1 = (1 + n_3) \frac{2n_3}{n_1 + in_2} \,, \\ y = \frac{1 + 2n_3}{n_1 + in_2} \,, \quad x_4 = \frac{i}{n_1 + in_2} - (1 + n_3) \frac{2n_3}{n_1 + in_2} \,. \tag{157}$$

Thus, we have found two massless solutions for spin 3/2 field, and found expansions of them in linear combinations terms of eigenstates with helicities $\sigma = \pm 1/2, \pm 3/2$.

IX. CONCLUSION

In the paper, solutions in the form of plane waves for a massive spin 3/2 particle are examined. The wave equation gives 4 algebraic equations for 8 unknown variables, which assumes existence of 4 independent solutions. In order to relate the choice of independent solutions to quantum number of a physical operator, we study the problem of eigenvectors for relevant helicity operator. As expected, we get 4 eigenvalues, $\sigma = \pm 1/2, \pm 3/2$. The values $\sigma = \pm 1/2$ turn out to have double multiplicity, this leads to existence of two different eigenstates both for $\sigma = -1/2$ and $\sigma = +1/2$. It is shown that the states with the values represent exact solutions of the wave equation. However, the double degenerate states separately do not. It is shown that exact solutions of the wave equation can be constructed in the form of special linear combinations of those. Thus, there constructed a complete system of exact solution for a massive spin 3/2 particle in momentum-helicity basis.

Initial wave equation for vector bispinor $\Psi_a(x)$, describing a massless spin 3/2 particle in Rarita-Sscwinger form, is transformed to a new basis $\tilde{\Psi}_a(x)$, in which the gauge symmetry in the theory becomes evident: there exist solutions in the form of 4-gradient of an arbitrary bispinor $\tilde{\Psi}_a^0(x) = \partial_a \Psi(x)$. For 16-component equation in this new basis, two independent solution are constructed explicitly, which do not contain gauge constituents. Previously, in the basis $\Psi_a(x)$, the eigenvalue problem for helicity operator of the spin 3/2 particle was solved, and six types of eigenstates were found; there are possible eigenvalues $\sigma = \pm 1/2, \pm 3/2$; the states with $\sigma = \pm 1/2$ are doubly degenerate. Massless solutions are transformed to initial Rarita-Schwinger basis, after that they are decomposed into linear combinations of helicity states, the relevant formulas contain terms related to helicities $\sigma = \pm 1/2$ and $\sigma = \pm 3/2$.

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