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Spin 1 Particle with Anomalous Magnetic Moment in the External Coulomb Field

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We study the quantum mechanical problem of a vector particle with anomalous magnetic moment in the external Coulomb field. From the relativistic Duffin-Kemmer-Petiau equation, we derive the system of 10 radial equations which can be split into two systems of 4 and 6 equations for states with parities $P = (-1)^{j+1}$ and $P = (-1)^j$, respectively. The interaction terms which are due to anomalous magnetic moment are present only in the system corresponding to states with parity $P = (-1)^j$. The non-relativistic approximation states with minimal value $j = 0$ are described by a second order equation of double confluent Heun type. By imposing the known transcendency condition, we derive the energy spectrum of the structure, $E = -\text{const}/n^2$. The numerical values for energies seem to be physically reasonable, though they do not depend on the anomalous magnetic moment. For states with $j = 1, 2, ...$, we obtain a system of two second order linked differential equations for radial functions. Its Frobenius solutions are constructed, and convergence of the involved power series with 8- and respectively 9-terms recurrence relations, are studied.

We additionally apply the geometrical method based on the use of KCC-invariants. The first and the second invariants are calculated. It is shown that the different branches of the solutions converge near the singular points $\infty, -\Gamma/2$, and may either converge or diverge near the singular point $r = 0$. This correlates with the expected behavior of solutions for bound states. The explicit Lagrangians related to the geometrical problem are determined. It is shown that the Lagrangians have the arbitrariness degree up to certain terms, which may be considered as specific gauge freedom.

I. INTRODUCTION

It is known that, in the framework of the theory of relativistic wave equations, one can consider the so-called non-minimal equations, which describe particles with additional electromagnetic characteristics. In particular, the equations for spin $S = 1/2$ and $S = 1$ particles with both electric charge and anomalous magnetic moment were studied extensively [1]–[12]. In the present paper we study the vector particle with anomalous magnetic moment in the external Coulomb field.

In the case of external Coulomb field, the equation for the vector particle is too complicated, even for the case of an ordinary particle without anomalous moment. This problem has not been solved completely yet. However, in the non-relativistic limit, the equation for an ordinary vector particle in the Coulomb field can be exactly solved. For this reason, in the present paper we investigate the non-relativistic problem for the particle with anomalous magnetic moment.

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II. SEPARATION OF THE VARIABLES

The initial equation has the following form[19]:

$$\left\{ i\beta^c \left[i(e^{\beta}_{(c)}\partial_\beta + \frac{1}{2}j^{ab}\gamma_{abc}(x)) \ - e' A_c \right] + \lambda \frac{e}{M} F_{\alpha\beta}(x) P j^{\alpha\beta}(x) - M \right\} \Psi = 0 \, ; \tag{1}$$

where the free parameter λ is dimensionless, P is the projective operator which selects the vector component from the 10-components function

$$P = \left| \begin{array}{cc} I_4 & 0 \\ 0 & 0 \end{array} \right| ;$$

we use the following notations:

$$M = \frac{mc}{\hbar}, \quad e' = \frac{e}{c\hbar}, \quad \Gamma = \lambda \frac{4\alpha}{M}, \quad \alpha = \frac{e^2}{\hbar c} = \frac{1}{137} \, . \tag{2}$$

In the spherical tetrad [14], the equation (1) takes the form

$$\left[\beta^0 (i\partial_t + \frac{\alpha}{r}) + i(\beta^3 \partial_r + \frac{1}{r}(\beta^1 j^{31} + \beta^2 j^{32})) + \frac{1}{r}\Sigma_{\theta,\phi} + \frac{\Gamma}{r^2} P j^{03} - M \right] \Phi = 0, \tag{3}$$

where the angular operator is determined by the equality:

$$\Sigma_{\theta,\phi} = i \ \beta^1 \partial_\theta + \beta^2 \frac{i\partial_\phi + i j^{12}\cos\theta}{\sin\theta} \, . \tag{4}$$

Relative to the used basis, the explicit form of the total angular momentum operator components is given by the formulas [14]:

$$j_1 = l_1 + \frac{\cos\phi}{\sin\theta} i j^{12}, \quad j_2 = l_2 + \frac{\sin\phi}{\sin\theta} i j^{12}, \quad j_3 = l_3 \, , \quad j^{12} = \beta^1\beta^2 - \beta^2\beta^1 .$$

We shall further use the wave function and the Duffin-Kemmer matrices in cyclic representation [14]; the matrix ij^{12} has the diagonal structure

$$ij^{12} = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & t_3 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_3 \end{array} \right| , \quad t_3 = \left| \begin{array}{ccc} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right| .$$

The form of the projective operator P does not change under transition from Cartesian basis to the cyclic one.

The system of radial equations for an ordinary vector particle in the Coulomb field is well-known [12]. To obtain the generalized system for the vector particle with anomalous magnetic moment, it suffices to specify the additional term in the equation:

$$\frac{\Gamma}{r^2} P j^{03} = \frac{\Gamma}{r^2} P (\beta^0\beta^3 - \beta^3\beta^0) \, . P \, j^{03} = \left| \begin{array}{cccccccccc} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right| . \tag{5}$$

The structure of the 10-components wave function for the vector particle with quantum numbers ϵ, j, m is the following

$$\Psi(x) = \{ \ f_0(x), \ \vec{f}(x), \ \vec{E}(x), \vec{H}(x) \ \} \, ,$$

$$\Phi_0(x) = e^{-i\epsilon t} f_0(r) D_0 \, , \quad \vec{\Phi}(x) = e^{-i\epsilon t} \left| \begin{array}{c} f_1(r) D_{-1} \\ f_2(r) D_0 \\ f_3(r) D_{+1} \end{array} \right| ,$$

$$\vec{E}(x) = e^{-i\epsilon t} \left| \begin{array}{c} E_1(r) D_{-1} \\ E_2(r) D_0 \\ E_3(r) D_{+1} \end{array} \right| , \quad \vec{H}(x) = e^{-i\epsilon t} \left| \begin{array}{c} H_1(r) D_{-1} \\ H_2(r) D_0 \\ H_3(r) D_{+1} \end{array} \right| , \tag{6}$$

where D stands for the Wigner functions [14]: $D_\sigma = D^j_{-m,\sigma}(\phi,\theta,0),\ \sigma = 0,-1,+1$.

After performing the needed calculations, one finds the system of radial equations

$$
\begin{aligned}
(\frac{d}{dr}+\frac{2}{r})\,E_2-\frac{\nu}{r}\,(E_1+E_3)-\frac{\Gamma}{r^2}f_2 &= m\,f_0\,,\\
+i\,(\epsilon+\frac{\alpha}{r})\,E_1\ \ +i\,(\frac{d}{dr}+\frac{1}{r})\,H_1+i\frac{\nu}{r}\,H_2 &= mf_1\,,\\
+i\,(\epsilon+\frac{\alpha}{r})\,E_2-i\frac{\nu}{r}\,(H_1-H_3)-\frac{\Gamma}{r^2}f_0 &= m\,f_2\,,\\
+i\,(\epsilon+\frac{\alpha}{r})\,E_3\ \ -i\,(\frac{d}{dr}+\frac{1}{r})\,H_3-i\frac{\nu}{r}\,H_2 &= m\,f_3\,;\\
-i\,(\epsilon+\frac{\alpha}{r})f_1+\frac{\nu}{r}f_0 = mE_1, -i(\epsilon+\frac{\alpha}{r})f_2-\frac{d}{dr}f_0 &= mE_2,\\
-i(\epsilon+\frac{\alpha}{r})f_3+\frac{\nu}{r}f_0 = mE_3, -i\,(\frac{d}{dr}+\frac{1}{r})f_1-i\frac{\nu}{r}f_2 &= mH_1,\\
+i\frac{\nu}{r}(f_1-f_3) = mH_2, +i(\frac{d}{dr}+\frac{1}{r})f_3+i\frac{\nu}{r}f_2 &= mH_3.
\end{aligned}
\tag{7}
$$

We additionally diagonalize the spatial inversion operator $\hat{\Pi}$. In the Cartesian basis β^a, this operator takes the form:

$$
\hat{\Pi} = \left|\begin{array}{cccc} 1 & 0 & 0 & 0\\ 0 & -I & 0 & 0\\ 0 & 0 & -I & 0\\ 0 & 0 & 0 & +I \end{array}\right| \hat{P}\,,\quad \hat{P}\Psi(\vec{r}) = \Psi(-\vec{r})\,. \tag{8}
$$

After transition to the spherical basis, we obtain an alternative representation of this operator:

$$
\hat{\Pi}' = \left|\begin{array}{cccc} 1 & 0 & 0 & 0\\ 0 & \Pi_3 & 0 & 0\\ 0 & 0 & \Pi_3 & 0\\ 0 & 0 & 0 & -\Pi_3 \end{array}\right| \hat{P}\,,\quad \Pi_3 = \left|\begin{array}{ccc} 0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0 \end{array}\right|\,. \tag{9}
$$

The spectral equation $\hat{\Pi}'\Psi = P\Psi$ has two types of solutions:

$$
\begin{array}{ll}
P = (-1)^{j+1}, & f_0 = 0\,,\ f_3 = -f_1,\ f_2 = 0,\ E_3 = -E_1,\ E_2 = 0,\ H_3 = H_1;\\
P = (-1)^j, & f_3 = +f_1,\ E_3 = +E_1,\ H_3 = -H_1,\ H_2 = 0.
\end{array}
\tag{10}
$$

For states with $P = (-1)^{j+1}$, we obtain four equations:

$$
\begin{aligned}
i\left(\epsilon+\frac{\alpha}{r}\right)E_1+i\left(\frac{d}{dr}+\frac{1}{r}\right)H_1+i\frac{\nu}{r}H_2 = mf_1\,,\\
-i\left(\epsilon+\frac{\alpha}{r}\right)f_1 = mE_1\,,\ -i\left(\frac{d}{dr}+\frac{1}{r}\right)f_1 = mH_1\,,\ 2i\frac{\nu}{r}f_1 = mH_2\,.
\end{aligned}
\tag{11}
$$

Here the anomalous magnetic moment does not manifest itself in any way in the external Coulomb field. The last 4-equations system allows the following exact solution for the main function f_1:

$$
\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+(\epsilon+\frac{\alpha}{r})^2-\frac{j(j+1)}{r^2}\right)f_1 = 0\,. \tag{12}
$$

The same equation arises in the theory of the scalar particle, in the presence of the Coulomb field. Its exact solutions and the corresponding energy spectrum are well-known.

For states with parity $P = (-1)^j$, we have six equations:

$$
\begin{aligned}
-(\frac{d}{dr}+\frac{2}{r})E_2-2\frac{\nu}{r}E_1-\frac{\Gamma}{r^2}f_2 = mf_0,\ i(\epsilon+\frac{\alpha}{r})E_1+i(\frac{d}{dr}+\frac{1}{r})H_1 &= mf_1,\\
+i(\epsilon+\frac{\alpha}{r})E_2-2i\frac{\nu}{r}H_1-\frac{\Gamma}{r^2}f_0 = mf_2\,,\ -i(\epsilon+\frac{\alpha}{r})f_2-\frac{d}{dr}f_0 &= mE_2,\\
-i(\epsilon+\frac{\alpha}{r})f_1+\frac{\nu}{r}f_0 = mE_1\,,\ i(\frac{d}{dr}+\frac{1}{r})f_1+i\frac{\nu}{r}f_2 &= -mH_1\,.
\end{aligned}
\tag{13}
$$

III. THE CASE OF MINIMAL $j = 0$

For states with minimal $j = 0$, we should look for solutions of the different form

$$\Phi_0(x) = e^{-i\epsilon t}\, f_0(r)\,, \quad \vec{\Phi}(x) = e^{-i\epsilon t} \begin{vmatrix} 0 \\ f_2(r) \\ 0 \end{vmatrix}\,,$$
$$\vec{E}(x) = e^{-i\epsilon t} \begin{vmatrix} 0 \\ E_2(r) \\ 0 \end{vmatrix}\,, \quad \vec{H}(x) = e^{-i\epsilon t} \begin{vmatrix} 0 \\ H_2(r) \\ 0 \end{vmatrix}\,. \tag{14}$$

Here we have the following four radial equations

$$-(\frac{d}{dr} + \frac{2}{r})E_2 - \frac{\Gamma}{r^2} f_2 = m f_0\,, \quad +i(\epsilon + \frac{\alpha}{r})E_2 - \frac{\Gamma}{r^2} f_0 = m f_2\,,$$
$$-i(\epsilon + \frac{\alpha}{r}) f_2 - \frac{d}{dr} f_0 = m E_2\,, \qquad H_2 = 0\,. \tag{15}$$

Excluding the variable E_2, we find the system for the unknown functions f_0, f_2:

$$i\left((\epsilon + \frac{\alpha}{r})\frac{d}{dr} + \frac{2\epsilon}{r} + \frac{\alpha + i\Gamma m}{r^2}\right) f_2 + \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - m^2\right) f_0 = 0\,,$$

$$f_2 = i\frac{r^2}{P(r)}\left((\epsilon + \frac{\alpha}{r})\frac{d}{dr} - \frac{i\Gamma m}{r^2}\right) f_0\,,$$

where $P(r) = (\epsilon^2 - m^2)r^2 + 2\epsilon\alpha r + \alpha^2$. The function f_2 can be excluded:

$$\frac{d^2 f}{dx^2} + (-\frac{2x}{P} + \frac{2\alpha^2 x + 6E\alpha x^2 + 4E^2 x^3}{P^2}$$

$$+ \frac{(2E^2 - 2E^4)x^5 + (4E\alpha - 6E^3\alpha)x^4 + (2\alpha^2 - 6E^2\alpha^2)x^3 - 2E\alpha^3 x^2}{P^3})\frac{df}{dx}$$

$$+(\frac{x^2}{P} + \frac{-2\,ixE\gamma + \gamma^2 - i\alpha\gamma}{P^2}$$

$$+ \frac{(-2iE\gamma + 2iE^3\gamma)x^3 + (-2i\alpha\gamma + 4iE^2\alpha\gamma)x^2 + 2iE\alpha^2\gamma x}{P^3})f = 0\,, \tag{16}$$

where the following dimensionless variables x, E and γ are used:

$$x = mr, \quad E = \frac{\epsilon}{m}, \quad \gamma = m\Gamma, \quad \alpha = \frac{1}{137}\,. \tag{17}$$

We note that the obtained equation has a complicated set of singular points. In the next section we will derive its much simpler non-relativistic analogue.

IV. THE NON-RELATIVISTIC APPROXIMATION. THE $j = 0$ CASE

In the system

$$\frac{1}{m}\left(-(\frac{d}{dr} + \frac{2}{r})E_2 - \frac{\Gamma}{r^2} f_2\right) = f_0\,,$$
$$+i(\epsilon + \frac{\alpha}{r})E_2 - \frac{\Gamma}{r^2} f_0 = m f_2\,, \quad -i(\epsilon + \frac{\alpha}{r}) f_2 - \frac{d}{dr} f_0 = m E_2$$

we shall first exclude the non-dynamical variable $f_0(r)$:

$$+i(\epsilon+\frac{\alpha}{r})E_2-\frac{\Gamma}{mr^2}\left(-(\frac{d}{dr}+\frac{2}{r})E_2-\frac{\Gamma}{r^2}f_2\right)=mf_2\,,$$
$$-i(\epsilon+\frac{\alpha}{r})f_2-\frac{1}{m}\frac{d}{dr}\left(-(\frac{d}{dr}+\frac{2}{r})E_2-\frac{\Gamma}{r^2}f_2\right)=mE_2\,.$$

Then we introduce the the big and the small components:

$$f_2=(B_2+M_2),\quad iE_2=(B_2-M_2)\,. \tag{18}$$

We simultaneously separate the rest energy by means of the formal substitution $\epsilon \Longrightarrow m+E$, where E is the non-relativistic energy. This results in

$$(E+\frac{\alpha}{r})(B_2-M_2)-\frac{\Gamma}{mr^2}[i(\frac{d}{dr}+\frac{2}{r})(B_2-M_2)-\frac{\Gamma}{r^2}(B_2+M_2)]=2mM_2\,,$$

$$(E+\frac{\alpha}{r})(B_2+M_2)-\frac{1}{m}\frac{d}{dr}[-(\frac{d}{dr}+\frac{2}{r})(B_2-M_2)-\frac{i\Gamma}{r^2}(B_2+M_2)]=-2mM_2\,.$$

To obtain the equation for the big component B_2, we sum these equations and then neglect the small component M_2:

$$2(E+\frac{\alpha}{r})B_2-\frac{\Gamma}{mr^2}\left(i(\frac{d}{dr}+\frac{2}{r})-\frac{\Gamma}{r^2}\right)B_2+\frac{1}{m}\frac{d}{dr}\left(\frac{d}{dr}+\frac{2}{r}+\frac{i\Gamma}{r^2}\right)B_2=0.$$

Taking into account that from the physical point of view the parameter Γ is imaginary, and making the substitution $i\Gamma \Longrightarrow \Gamma$, we find (while changing the notation as $B_2(r)=R(r)$):

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(2m(E+\frac{\alpha}{r})-\frac{2}{r^2}-\frac{4\Gamma}{r^3}-\frac{\Gamma^2}{r^4}\right)R=0\,. \tag{19}$$

The equation (19) has two irregular singular points $r=0$ and $r=\infty$, both of them having rank 2. This equation belongs to the double confluent Heun type. The local Frobenius solutions in the vicinity of the point $r=0$ are constructed as

$$R=e^{Ar}r^Be^{\frac{C}{r}}f(r)\,. \tag{20}$$

For the function $f(r)$ we obtain the equation

$$f''+(2A+\frac{2B+2}{r}-\frac{2C}{r^2})f'+\left(2mE+A^2+\frac{2AB+2m\alpha+2A}{r}+\right.$$
$$\left.+\frac{B^2+B-2AC-2}{r^2}+\frac{-4\Gamma-2BC}{r^3}+\frac{-\Gamma^2+C^2}{r^4}\right)f=0\,.$$

Let us impose the restrictions

$$A=\pm\sqrt{-2mE}\ \ (E<0)\,,\quad C=\pm\Gamma\,,\quad B=-\frac{2\Gamma}{C}=\mp 2\,;$$

to describe the bounded states one should use the following parameters:

$$\begin{array}{ll}\Gamma>0, & A=-\sqrt{-2mE}\,,\quad C=-\Gamma\,,\quad B=+2\,;\\ \Gamma<0, & A=-\sqrt{-2mE}\,,\quad C=+\Gamma\,,\quad B=-2\,.\end{array} \tag{21}$$

Taking into account (21), the previous equation reduces to

$$f''+\left(2A+\frac{2B+2}{r}-\frac{2C}{r^2}\right)f'+\left(\frac{2AB+2m\alpha+2A}{r}+\frac{B^2+B-2AC-2}{r^2}\right)f=0\,;$$

we notice two cases which differ, depending on the sign of Γ:

$$\Gamma > 0, \qquad f'' + \left(-2\sqrt{-2mE} + \frac{6}{r} + \frac{2\Gamma}{r^2}\right) f' + \left(\frac{-6\sqrt{-2mE} + 2m\alpha}{r} + \frac{4 - 2\Gamma\sqrt{-2mE}}{r^2}\right) f = 0\,; \tag{22}$$

$$\Gamma < 0, \qquad f'' + \left(-2\sqrt{-2mE} - \frac{2}{r} - \frac{2\Gamma}{r^2}\right) f' + \left(\frac{+2\sqrt{-2mE} + 2m\alpha}{r} + \frac{2\Gamma\sqrt{-2mE}}{r^2}\right) f = 0\,. \tag{23}$$

Both these equations have the same mathematical structure:

$$f'' + (a + \frac{a_1}{r} + \frac{a_2}{r^2})f' + (\frac{b_1}{r} + \frac{b_2}{r^2})f = 0\,. \tag{24}$$

The solutions of eq. (24) are constructed as power series with 3-terms recurrence relations:

$$k \geq 2, \quad [a(k-1) + b_1]\, c_{k-1} + [k(k-1) + a_1 k + b_2]\, c_k + a_2(k+1)\, c_{k+1} = 0\,,$$

or shortly

$$P_{k-1}c_{k-1} + P_k c_k + P_{k+1}c_{k+1} = 0\,, \tag{25}$$

where

$$P_{k-1} = a(k-1) + b_1\,, \quad P_k = k(k-1) + a_1 k + b_2\,, \quad P_{k+1} = a_2(k+1)\,.$$

Dividing the relation (25) by $c_{k-1}k^2$ and tending $k \to \infty$:

$$\frac{1}{k^2}[a(k-1) + b_1] + \frac{1}{k^2}[k(k-1) + a_1 k + b_2]\frac{c_k}{c_{k-1}} + \frac{1}{k^2}a_2(k+1)\frac{c_{k+1}}{c_k}\frac{c_k}{c_{k-1}} = 0\,,$$

we obtain the algebraic equation that defines the convergence radius:

$$r = 0, \qquad \Longrightarrow \qquad R_{\text{conv}} = \frac{1}{|r|} = \infty\,. \tag{26}$$

Let us write down the expression for the coefficients in (25):

$$\begin{gathered} \Gamma > 0, \quad P_{k-1} = -2\sqrt{-2mE}\,(k-1) - 6\sqrt{-2mE} + 2m\alpha\,, \\ P_k = k(k-1) + 6k + 4 - 2\sqrt{-2mE}\,\Gamma\,, \quad P_{k+1} = 2\Gamma(k+1)\,; \end{gathered} \tag{27}$$

$$\begin{gathered} \Gamma < 0, \quad P_{k-1} = -2\sqrt{-2mE}\,(k-1) + 2\sqrt{-2mE} + 2m\alpha\,, \\ P_k = k(k-1) - 2k + 2\sqrt{-2mE}\,\Gamma\,, \quad P_{k+1} = -2\Gamma(k+1)\,. \end{gathered} \tag{28}$$

To get some quantization rule, we apply the condition to determine transcendental Heun functions:

$$\Gamma > 0\,, \quad P_{k-1} = -2\sqrt{-2mE}(k-1) - 6\sqrt{-2mE} + 2m\alpha = 0\,,$$

$$\Gamma < 0\,, \quad p_{k-1} = -2\sqrt{-2mE}\,(k-1) + 2\sqrt{-2mE} + 2m\alpha \; = 0\,.$$

From this, we find two following different formulas for energy values:

$$\Gamma > 0, \qquad E = -\frac{m\alpha^2}{2}\frac{1}{(k+2)^2} \quad k \geq 2\,; \tag{29}$$

$$\Gamma < 0, \qquad E = -\frac{m\alpha^2}{2}\frac{1}{(k-2)^2}\,, \quad k > 2\,. \tag{30}$$

These formulas seem to be physically interpretable; however, they hardly describe correct energy spectra, since they do not depend on the parameter of anomalous magnetic moment Γ.

V. NON-RELATIVISTIC EQUATIONS FOR $j = 1, 2, 3, ...$

Let us start with the relativistic equations:

$$-(\frac{d}{dr}+\frac{2}{r})E_2 - 2\frac{\nu}{r}E_1 - \frac{\Gamma}{r^2 f_2} = mf_0\,, \qquad i(\epsilon+\frac{\alpha}{r})E_1 + i(\frac{d}{dr}+\frac{1}{r})H_1 = mf_1\,,$$

$$+i(\epsilon+\frac{\alpha}{r})E_2 - 2i\frac{\nu}{r}H_1 - \frac{\Gamma}{r^2}f_0 = mf_2\,, \qquad -i(\epsilon+\frac{\alpha}{r})f_2 - \frac{d}{dr}f_0 = mE_2\,,$$

$$-i(\epsilon+\frac{\alpha}{r})f_1 + \frac{\nu}{r}f_0 = mE_1\,, \qquad +i(\frac{d}{dr}+\frac{1}{r})f_1 + i\frac{\nu}{r}f_2 = -mH_1\,.$$

By excluding the non-dynamical variables f_0, H_1, we get four equations

$$+i(\epsilon+\frac{\alpha}{r})E_1 - i(\frac{d}{dr}+\frac{1}{r})\frac{1}{m}\left[i(\frac{d}{dr}+\frac{1}{r})f_1 + i\frac{\nu}{r}f_2\right] = mf_1\,,$$

$$-i(\epsilon+\frac{\alpha}{r})f_1 - \frac{\nu}{r}\frac{1}{m}\left[(\frac{d}{dr}+\frac{2}{r})E_2 + 2\frac{\nu}{r}E_1 + \frac{\Gamma}{r^2}f_2\right] = mE_1\,,$$

$$i(\epsilon+\frac{\alpha}{r})E_2 + 2i\frac{\nu}{r}\frac{1}{m}\left[i(\frac{d}{dr}+\frac{1}{r})f_1 + i\frac{\nu}{r}f_2\right] + \frac{\Gamma}{r^2}\frac{1}{m}\left[(\frac{d}{dr}+\frac{2}{r})E_2 + 2\frac{\nu}{r}E_1 + \frac{\Gamma}{r^2}f_2\right] = mf_2\,,$$

$$-i(\epsilon+\frac{\alpha}{r})f_2 + \frac{d}{dr}\frac{1}{m}\left[(\frac{d}{dr}+\frac{2}{r})E_2 + 2\frac{\nu}{r}E_1 + \frac{\Gamma}{r^2}f_2\right] = mE_2\,.$$

The big and the small components are determined by the formulas

$$f_1 = (\Psi_1+\psi_1), \quad iE_1 = (\Psi_1-\psi_1), \qquad f_2 = (\Psi_2+\psi_2), \quad iE_2 = (\Psi_2-\psi_2). \tag{31}$$

Taking in mind eqs. (31) in the previous system (also separating the rest energy by the substitution $\epsilon = (m+E)$), we derive

$$(E+\frac{\alpha}{r})(\Psi_1-\psi_1) + (\frac{d}{dr}+\frac{1}{r})\frac{1}{m}[(\frac{d}{dr}+\frac{1}{r})(\Psi_1+\psi_1) + \frac{\nu}{r}(\Psi_2+\psi_2)] = 2m\psi_1\,,$$

$$(E+\frac{\alpha}{r})(\Psi_1+\psi_1) - \frac{\nu}{r}\frac{1}{m}[(\frac{d}{dr}+\frac{2}{r})(\Psi_2-\psi_2) + 2\frac{\nu}{r}(\Psi_1-\psi_1) + \frac{i\Gamma}{r^2}(\Psi_2+\psi_2)] = -2m\psi_1\,,$$

$$(E+\frac{\alpha}{r})(\Psi_2-\psi_2) - 2\frac{\nu}{r}\frac{1}{m}[(\frac{d}{dr}+\frac{1}{r})(\Psi_1+\psi_1) + \frac{\nu}{r}(\Psi_2+\psi_2)]+$$

$$+\frac{\Gamma}{r^2}\frac{1}{m}[-i(\frac{d}{dr}+\frac{2}{r})(\Psi_2-\psi_2) - 2i\frac{\nu}{r}(\Psi_1-\psi_1) + \frac{\Gamma}{r^2}(\Psi_2+\psi_2)] = 2m\psi_2\,,$$

$$(E+\frac{\alpha}{r})(\Psi_2+\psi_2) + \frac{d}{dr}\frac{1}{m}[(\frac{d}{dr}+\frac{2}{r})(\Psi_2-\psi_2) + 2\frac{\nu}{r}(\Psi_1-\psi_1) + \frac{i\Gamma}{r^2}(\Psi_2+\psi_2)] = -2m\psi_2\,.$$

To get the non-relativistic equations for the big components Ψ_1 and Ψ_2, we sum the equations within each pair and neglect the small components. In this way we obtain

$$(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{\beta-\lambda r}{r}-\frac{2\nu^2}{r^2})\Psi_1 - \nu\frac{2r+\Gamma}{r^3}\Psi_2 = 0\,,$$

$$(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{\beta-\lambda r}{r}-\frac{2\nu^2}{r^2}-\frac{2}{r^2}-\frac{4\Gamma}{r^3}-\frac{\Gamma^2}{r^4})\Psi_2 - 2\nu\frac{2r+\Gamma}{r^3}\Psi_1 = 0\,. \tag{32}$$

In (32), we have performed the change $i\Gamma \to \Gamma$, and the following notations have been used:

$$2mE = -\lambda, \quad \lambda > 0, \quad 2m\alpha = \beta, \quad 2\nu^2 = j(j+1) \equiv L\,. \tag{33}$$

From the system (32) we can derive a 4-th order equation for the function $\Psi_1(r)$:

$$\frac{d^4}{dr^4}\Psi_1 + [-\frac{4}{2r+\Gamma} + \frac{10}{r}]\frac{d^3}{dr^3}\Psi_1 +$$

$$+[-2\lambda + \frac{2\Gamma\beta - 24}{\Gamma}\frac{1}{r} + \frac{22-2L}{r^2} - \frac{4\Gamma}{r^3} - \frac{\Gamma^2}{r^4} + \frac{48}{\Gamma(2r+\Gamma)} + \frac{8}{(2r+\Gamma)^2}]\frac{d^2}{dr^2}\Psi_1 +$$

$$+[+\frac{-8L + 64 - 10\Gamma^2\lambda - 4\Gamma\beta}{\Gamma^2 r} + \frac{4L - 24 + 8\Gamma\beta}{\Gamma r^2} + \frac{8-6L}{r^3} - \frac{8\Gamma}{r^4} - \frac{2\Gamma^2}{r^5} +$$

$$+\frac{4\Gamma^2\lambda + 16L - 128 + 8\Gamma\beta}{\Gamma^2(2r+\Gamma)} - \frac{32}{\Gamma(2r+\Gamma)^2}]\frac{d}{dr}\Psi_1 +$$

$$+[\lambda^2 + \frac{16\Gamma^2\lambda + 64L + 32\Gamma\beta - 2\beta\lambda\Gamma^3}{\Gamma^3 r} + \frac{-10\Gamma^2\lambda - 24L - 12\Gamma\beta + \beta^2\Gamma^2 + 2\lambda L\Gamma^2}{\Gamma^2 r^2} +$$

$$+\frac{4\Gamma^2\lambda + 4\Gamma\beta + 8L - 2\Gamma\beta L}{\Gamma r^3} + \frac{-4\Gamma\beta + L^2 + \Gamma^2\lambda - 4L}{r^4} - \frac{\Gamma^2\beta}{r^5} +$$

$$+\frac{-32\Gamma^2\lambda - 128L - 64\Gamma\beta}{\Gamma^3(2r+\Gamma)} + \frac{-32L - 8\Gamma^2\lambda - 16\Gamma\beta}{\Gamma^2(2r+\Gamma)^2}]\Psi_1 = 0.$$

The equation structure may be briefly written as

$$\begin{aligned} &\frac{d^4}{dr^4}\Psi_1 + \left[-\frac{4}{2r+\Gamma} + \frac{10}{r}\right]\frac{d^3}{dr^3}\Psi_1 + \\ &+\left[-2\lambda + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \frac{a_4}{r^4} + \frac{a_5}{2r+\Gamma} + \frac{a_6}{(2r+\Gamma)^2}\right]\frac{d^2}{dr^2}\Psi_1 + \\ &+\left[\frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \frac{b_4}{r^4} + \frac{b_5}{r^5} + \frac{b_6}{2r+\Gamma} + \frac{b_7}{(2r+\Gamma)^2}\right]\frac{d}{dr}\Psi_1 + \\ &+\left[\lambda^2 + \frac{c_1}{r} + \frac{c_2}{r^2} + \frac{c_3}{r^3} + \frac{c_4}{r^4} + \frac{c_5}{r^5} + \frac{c_6}{2r+\Gamma} + \frac{c_7}{(2r+\Gamma)^2}\right]\Psi_1 = 0. \end{aligned} \tag{34}$$

In the neighborhood of the regular singular point $r = -\Gamma/2$, eq. (34) simplifies:

$$\left(\frac{d^4}{dr^4} - \frac{4}{2r+\Gamma}\frac{d^3}{dr^3} + \frac{a_6}{(2r+\Gamma)^2}\frac{d^2}{dr^2} + \frac{b_7}{(2r+\Gamma)^2}\frac{d}{dr} + \frac{c_7}{(2r+\Gamma)^2}\right)\Psi_1 = 0,$$

and its solutions have the form

$$\Psi_1 = (2r+\Gamma)^s, \qquad s = 0,\ -1,\ -3,\ -4\,. \tag{35}$$

Only at $s = 0$, the corresponding solution behaves regularly near the point $r = -\Gamma/2$.

The point $r = 0$ is an irregular singular point of the rank 2. Therefore, the local Frobenius solutions should be searched of the form

$$\Psi_1(r) = e^{Dr} r^A e^{B/r} f(r)\,. \tag{36}$$

The differential equation for the function $f(r)$ has the following structure:

$$f'''' + (...)f''' + (...)f'' + (...)f' + (\lambda^2 - 2\,\lambda\,D^2 + D^4 + ...)f = 0\,. \tag{37}$$

The explicit form of this equation is too complicated, so we omit it. We shall study it for physically relevant values of the parameters.

The parameter D is fixed by the requirement

$$\lambda^2 - 2\,\lambda\,D^2 + D^4 = 0, \quad (D^2-\lambda)^2 = 0, \quad D = \pm\sqrt{\lambda} = \pm\sqrt{-2mE} = \pm\sqrt{-2\epsilon}\,. \tag{38}$$

Since the coefficient of the term $\frac{1}{r^8} f(r)$ should vanish, the following restriction on the parameter B is imposed:

$$B^2(a_4+B^2) = 0 \quad \Longrightarrow \quad B_1 = 0, B_2 = +\Gamma, B_3 = -\Gamma\,, \tag{39}$$

where we note that one of the roots is double degenerate.

Since $B_1 = 0$, the coefficient of $\frac{1}{r^7} f(r)$ identically vanishes, so we impose the restriction of vanishing on the coefficient of $\frac{1}{r^6} f(r)$:

$$a_4 A^2 \Gamma + b_5 A \Gamma - a_4 A \Gamma = 0 \qquad \Longrightarrow \quad A_1 = 0,\ A_2 = -1\,. \tag{40}$$

While $B = \pm\Gamma$, from the requirement of zero-coefficient for r^{-7}, we obtain the equation for the parameter A:

$$2\,a_4 A - a_3 B + b_5 + 4\,AB^2 - 2\,a_4 - 2\,B^2 = 0\,,$$

or,

$$A(2a_4 + 4B^2) = a_3 B - b_5 + (2a_4 + 2B^2)\,.$$

Taking into account that

$$a_4 = -\Gamma^2, \quad a_3 = -4\gamma, \quad b_5 = -2\Gamma^2, \quad B = \pm\Gamma,$$

one finds $2\Gamma^2 A = -4\Gamma(\pm\Gamma) + 2\Gamma^2 + 0$, or,

$$B = +\Gamma, A_3 = -1\,; \quad B = -\Gamma, A_4 = +3\,. \tag{41}$$

Therefore, we have four independent solutions for the given parameters D, A, B (we consider only the case of negative D):

$$(I, II)\ \ D = -\sqrt{-2\epsilon}, B = 0, A_1 = 0,\ \ A_2 = -1, \quad \begin{array}{l} \Psi_1 = e^{Dr} f_1(r), \\ \Psi_2 = e^{Dr} \frac{1}{r} f_2(r)\,; \end{array} \tag{42}$$

$$(III)\ \ D = -\sqrt{-2\epsilon}, B = +\Gamma, A_3 = -1, \quad \Psi_3 = e^{Dr} \frac{1}{r} e^{+\Gamma/r} f_3(r)\,; \tag{43}$$

$$(IV)\ \ D = -\sqrt{-2\epsilon}, B = -\Gamma,\ \ A_4 = +3, \quad \Psi_4 = e^{Dr} r^3 e^{-\Gamma/r} f_4(r). \tag{44}$$

Let us study the solution of type I:

$$D = -\sqrt{\lambda}, \quad B = 0, \quad A_1 = 0\,, \tag{45}$$

the equation (37) takes the form

$$\frac{d^4}{dr^4} f + \left(P_0 + \frac{P_1}{r} + \frac{P_2}{2r+\Gamma} \right) \frac{d^3}{dr^3} f +$$

$$+ \left(Q_0 + \frac{Q_1}{r} + \frac{Q_2}{r^2} + \frac{Q_3}{r^3} + \frac{Q_4}{r^4} + \frac{Q_5}{2r+\Gamma} + \frac{Q_6}{(2r+\Gamma)^2} \right) \frac{d^2}{dr^2} f +$$

$$+ \left(\frac{M_1}{r} + \frac{M_2}{r^2} + \frac{M_3}{r^3} + \frac{M_4}{r^4} + \frac{M_5}{r^5} + \frac{M_6}{2r+\Gamma} + \frac{M_7}{(2r+\Gamma)^2} \right) \frac{d}{dr} f +$$

$$+\left(\frac{N_1}{r}+\frac{N_2}{r^2}+\frac{N_3}{r^3}+\frac{N_4}{r^4}+\frac{N_5}{r^5}+\frac{N_6}{2r+\Gamma}+\frac{N_7}{(2r+\Gamma)^2}\right)f=0\,. \tag{46}$$

The solutions for the functions are constructed as power series:

$$f=\sum_{n=0}^{\infty}d_n r^n,\quad f'=\sum_{n=1}^{\infty}n d_n r^{n-1},\quad f''=\sum_{n=2}^{\infty}n(n-1)d_n r^{n-2},$$

$$f'''=\sum_{n=3}^{\infty}n(n-1)(n-2)d_n r^{n-3},\quad f''''=\sum_{n=4}^{\infty}n(n-1)(n-2)(n-3)d_n r^{n-4}.$$

Equating the coefficients of the variable which have the same power r^n, we find the 8-terms recurrence relation: $n\geq 6$,

$$[\,4\,N_1+2\,N_6\,]\,d_{n-6}+$$

$$+[\,4Q_0(n-5)(n-6)+(4\,M_1+2\,M_6)(n-5)+(4\,\Gamma\,N_1+4\,N_2+\Gamma)\,N_6+N_7)\quad d_{n-5}+$$

$$+[\,4P_0(n-4)(n-5)(n-6)+(4\,Q_1+4\,\Gamma\,Q_0+2\,Q_5)(n-4)(n-5)+$$

$$+(4\,\Gamma\,M_1+4\,M_2+\Gamma\,M_6+M_7)(n-4)+(\Gamma^2 N_1+4\,N_3+4\,\Gamma\,N_2)\,]\,d_{n-4}+$$

$$[\,4(n-3)(n-4)(n-5)(n-6)+(4\,\Gamma\,P_0+4\,P_1+2\,P_2)(n-3)(n-4)(n-5)+$$

$$+(\Gamma\,Q_5+\Gamma^2 Q_0+4\,\Gamma\,Q_1+4\,Q_2+Q_6)(n-3)(n-4)+$$

$$+(4\,\Gamma\,M_2+\Gamma^2 M_1+4\,M_3)(n-3)+(4\,\Gamma\,N_3+\Gamma^2 N_2+4\,N_4)\,]\,d_{n-3}+$$

$$+[\,4\Gamma(n-2)(n-3)(n-4)(n-5)+(4\,\Gamma\,P_1+\Gamma^2 P_0+\Gamma\,P_2)(n-2)(n-3)(n-4)+$$

$$+(\Gamma^2 Q_1+4\,Q_3+4\,\Gamma\,Q_2)(n-2)(n-3)+(4\,M_4+4\,\Gamma\,M_3+\Gamma^2 M_2)(n-2)+$$

$$+(4\,\Gamma\,N_4+\Gamma^2 N_3+4\,N_5)\,]\,d_{n-2}+$$

$$+[\,\Gamma^2(n-1)(n-2)(n-3)(n-4)+\Gamma^2 P_1(n-1)(n-2)(n-3)+$$

$$+(4\,\Gamma\,Q_3+\Gamma^2 Q_2+4\,Q_4)(n-1)(n-2)+(\Gamma^2 M_3+4\,\Gamma\,M_4+4\,M_5)(n-1)+$$

$$+(\Gamma^2 N_4+4\,\Gamma\,N_5)\,]\,d_{n-1}+$$

$$+[\,(4\,\Gamma\,Q_4+\Gamma^2 Q_3)n(n-1)+(\Gamma^2 M_4+4\,\Gamma\,M_5)n+\Gamma^2 N_5\,]\,d_n+$$

$$+(n+1)\Gamma^2[\,nQ_4+M_5\,]\,d_{n+1}=0\,. \tag{47}$$

According to the Poincaré-Perrone method, we divide eq. (47) by $n^4 d_{n-6}$ and tend $n\to\infty$. In this way we arrive at an algebraic equation defining the possible convergence radius:

$$4r^3+4\Gamma r^4+\Gamma^2 r^5=0,\qquad r=0, r=-\frac{2}{\Gamma}\qquad\Longrightarrow\qquad R_{conv}=\frac{1}{|r|}=\infty,\quad\frac{\Gamma}{2}. \tag{48}$$

Now, let us investigate the solutions of the type II:

$$D = -\sqrt{\lambda},\ B = 0,\ A_2 = -1.$$

The equation (37) takes the from

$$\frac{d^4}{dr^4}f + \left(P_0' + \frac{P_1'}{r} + \frac{P_2'}{2r+\Gamma}\right)\frac{d^3}{dr^3}f +$$

$$+\left(Q_0' + \frac{Q_1'}{r} + \frac{Q_2'}{r^2} + \frac{Q_3'}{r^3} + \frac{Q_4'}{r^4} + \frac{Q_5'}{2r+\Gamma} + \frac{Q_6'}{(2r+\Gamma)^2}\right)\frac{d^2}{dr^2}f +$$

$$+\left(\frac{M_1'}{r} + \frac{M_2'}{r^2} + \frac{M_3'}{r^3} + \frac{M_4'}{r^4} + \frac{(M_5'=0)}{r^5} + \frac{M_6'}{2r+\Gamma} + \frac{M_7'}{(2r+\Gamma)^2}\right)\frac{d}{dr}f +$$

$$+\left(\frac{N_1'}{r} + \frac{N_2'}{r^2} + \frac{N_3'}{r^3} + \frac{N_4'}{r^4} + \frac{N_5'}{r^5} + \frac{N_6'}{2r+\Gamma} + \frac{N_7'}{(2r+\Gamma)^2}\right)f = 0\,. \tag{49}$$

In contrast to the equation (46), here $M_5' = 0$. The solutions of this type are not relevant for bound states, since they diverge near the point $r = 0$ due to the term $r^A = r^{-1}$.

For the solution of type III, the main equation has the form:

$$\frac{d^4}{dr^4}f + \left(P_0 + \frac{P_1}{r} + \frac{P_2}{r^2} + \frac{P_3}{2r+\Gamma}\right)\frac{d^3}{dr^3}f +$$

$$+\left(Q_0 + \frac{Q_1}{r} + \frac{Q_2}{r^2} + \frac{Q_3}{r^3} + \frac{Q_4}{r^4} + \frac{Q_5}{2r+\Gamma} + \frac{Q_6}{(2r+\Gamma)^2}\right)\frac{d^2}{dr^2}f +$$

$$+\left(\frac{M_1}{r} + \frac{M_2}{r^2} + \frac{M_3}{r^3} + \frac{M_4}{r^4} + \frac{M_5}{r^5} + \frac{M_6}{r^6} + \frac{M_7}{2r+\Gamma} + \frac{M_8}{(2r+\Gamma)^2}\right)\frac{d}{dr}f +$$

$$+\left(\frac{N_1}{r} + \frac{N_2}{r^2} + \frac{N_3}{r^3} + \frac{N_4}{r^4} + \frac{N_5}{r^5} + \frac{N_6}{r^6} + \frac{N_7}{2r+\Gamma} + \frac{N_8}{(2r+\Gamma)^2}\right)f = 0\,. \tag{50}$$

In this case, we construct the solutions as power series with 9-terms recurrence relations:

$n = 7, 8, 9...$

$$(2N_7 + 4\,N_1)\ d_{n-7} +$$

$$+[4Q_0(n-6)(n-7) + (4\,M_1 + 2\,M_7)(n-6) + (4\Gamma N_1 + N_8 + 4\,N_2 + \Gamma N_7)]\ d_{n-6} +$$

$$+[4P_0(n-5)(n-6)(n-7) + (4Q_1 + 4\Gamma Q_0 + 2\,Q_5)(n-5)(n-6) +$$

$$+(4\Gamma M_1 + \Gamma M_7 + 4\,M_2 + M_8)(n-5) + (\Gamma^2 N_1 + 4\Gamma N_2 + 4\,N_3)]\ d_{n-5} +$$

$$+[4(n-4)(n-5)(n-6)(n-7) + (4P_1 + 2P_3 + 4\Gamma\,P_0)(n-4)(n-5)(n-6) +$$

$$+(4\Gamma Q_1 + \Gamma^2 Q_0 + \Gamma Q_5 + 4\,Q_2 + Q_6)(n-4)(n-5) + (\Gamma^2 M_1 + 4\Gamma M_2 + 4\,M_3)(n-4) +$$

$$+(\Gamma^2 N_2 + 4\Gamma N_3 + 4\,N_4)]\ d_{n-4} + [4\Gamma(n-3)(n-4)(n-5)(n-6) +$$

$$+(\Gamma P_3 + \Gamma^2 P_0 + 4\,P_2 + 4\Gamma P_1)(n-3)(n-4)(n-5) + (4\,Q_3 + 4\Gamma\,Q_2 + \Gamma^2 Q_1)(n-3)(n-4)+$$

$$+(4\,M_4 + 4\Gamma M_3 + \Gamma^2 M_2)(n-3) + (4\Gamma N_4 + 4\,N_5 + \Gamma^2 N_3)]\;d_{n-3}+$$

$$+[\Gamma^2(n-2)(n-3)(n-4)(n-5) + (\Gamma^2 P_1 + 4\,P_2\Gamma)(n-2)(n-3)(n-4)+$$

$$+(\Gamma^2 Q_2 + 4\,Q_4 + 4\Gamma Q_3)(n-2)(n-3) + (4\,M_5 + 4\Gamma M_4 + \Gamma^2 M_3)(n-2)+$$

$$+(4\Gamma N_5 + 4\,N_6 + \Gamma^2 N_4)]\;d_{n-2} + [\Gamma^2 P_2(n-1)(n-2)(n-3)+$$

$$+(4\Gamma\,Q_4 + \Gamma^2 Q_3)(n-1)(n-2) + (4\Gamma M_5 + \Gamma^2 M_4 + 4\,M_6)(n-1)+$$

$$+(\Gamma^2 N_5 + 4\Gamma N_6)]\;d_{n-1} + [\Gamma^2 Q_4 n(n-1)+$$

$$+(\Gamma^2 M_5 + 4\Gamma M_6)n + \Gamma^2 N_6]\;d_n + \Gamma^2 M_6(n+1)\;d_{n+1} = 0\,. \tag{51}$$

Studying convergence of the series by the Poincaré-Perrone method, we find the following possible convergence radii:

$$4r^3 + 4\Gamma r^4 + \Gamma^2 r^5 = 0\,, \quad r = 0\,, r = -\frac{2}{\Gamma} \quad \Longrightarrow \quad R_{conv} = \frac{1}{|r|} = \infty, \quad \frac{|\Gamma|}{2}\,.$$

Finally, let us consider the last case IV at $B = -\Gamma, A = 3$. The equation (37) has the form:

$$\frac{d^4}{dr^4}f + \left(P_0^{'} + \frac{P_1^{'}}{r} + \frac{P_2^{'}}{r^2} + \frac{P_3^{'}}{2r+\Gamma}\right)\frac{d^3}{dr^3}f+$$

$$+\left(Q_0^{'} + \frac{Q_1^{'}}{r} + \frac{Q_2^{'}}{r^2} + \frac{Q_3^{'}}{r^3} + \frac{Q_4^{'}}{r^4} + \frac{Q_5^{'}}{2r+\Gamma} + \frac{Q_6^{'}}{(2r+\Gamma)^2}\right)\frac{d^2}{dr^2}f+$$

$$+\left(\frac{M_1^{'}}{r} + \frac{M_2^{'}}{r^2} + \frac{M_3^{'}}{r^3} + \frac{M_4^{'}}{r^4} + \frac{M_5^{'}}{r^5} + \frac{M_6^{'}}{r^6} + \frac{M_7^{'}}{2r+\Gamma} + \frac{M_8^{'}}{(2r+\Gamma)^2}\right)\frac{d}{dr}f+$$

$$+\left(\frac{N_1^{'}}{r} + \frac{N_2^{'}}{r^2} + \frac{N_3^{'}}{r^3} + \frac{N_4^{'}}{r^4} + \frac{N_5^{'}}{r^5} + \frac{N_6^{'}}{r^6} + \frac{N_7^{'}}{2r+\Gamma} + \frac{N_8^{'}}{(2r+\Gamma)^2}\right)f = 0\,.$$

The structure of this equation is the same as in the case III, so we have formally the same recurrence relations and the same convergence radii.

VI. KCC-GEOMETRICAL APPROACH TO THE PROBLEM UNDER CONSIDERATION

Now we consider the problem of spin 1 particle with anomalous magnetic moment in the external Coulomb fieldby applying the Kosambi–Cartan–Chern geometrical theory. This geometrical study of the relevant system of differential equations is based on the use of KCC-invariants [15–18].

In this approach, one considers a system of second order differential equations

$$\dot{y}^{\,i}(r) + 2Q^i(r,x,y) = 0\,, \tag{52}$$

which corresponds to the the Euler-Lagrange equations for some differential system associated to a Lagrangian function L. In (52), the symbol x^i designates so called coordinates, their derivatives in the argument r are $y^i = dx^i/dr = \dot{x}^i$, and the quantities Q_i are determined through some Lagrangian L, as follows

$$Q^i = \frac{1}{4}g^{il}\left(\frac{\partial^2 L}{\partial x^k \partial y^l}y^k - \frac{\partial L}{\partial x^l} + \frac{\partial^2 L}{\partial y^l \partial r}\right), \qquad g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}\,. \tag{53}$$

The first and the second invariants, $\varepsilon^i(r,x,y)$ and $P^i_{\ j}$, defined by

$$\varepsilon^i = \frac{\partial Q^i}{\partial y^j}\, y^j - 2Q^i,$$

$$P^i_{\ j} = 2\frac{\partial Q^i}{\partial x^j} + 2Q^s \frac{\partial^2 Q^i}{\partial y^j \partial y^s} - \frac{\partial^2 Q^i}{\partial y^j \partial x^s} y^s - \frac{\partial Q^i}{\partial y^s}\frac{\partial Q^s}{\partial y^j} - \frac{\partial^2 Q^i}{\partial y^j \partial r}. \tag{54}$$

The second invariant $P^i_{\ j}$ describes the Jacobi stability of the system. There is an analogy between the equations of Riemannian geodesic deviation, the ones governed by the second KCC-invariant:

$$\frac{D^2\xi^i}{Ds^2} = R^i_{kjl}\frac{dx^k}{ds}\frac{dx^l}{ds}\xi^j = -K^i_j\xi^j, \qquad \frac{D^2\xi^i}{Dr^2} = P^i_j\xi^j. \tag{55}$$

It is known that a pencil of geodesic curves which emerge from the same point r_0 converges (or diverges) if the real parts of all eigenvalues of the invariant $P^i_{\ j}$ are negative (or positive) ones.

We start with the radial system of two second-order differential equations (32) for two radial functions for the non-relativistic case. It should be emphasized that we shall follow the case of bound states, hence assuming $\nu = \sqrt{j(j+1)/2},\ \ j = 1, 2, 3, \ldots$.

We further apply the notations $x^i = \Psi_i(r),\ y^i = (d/dr)\Psi_i(r) = \dot{\Psi}_i(r)$. Then, by comparing equations (32) and (52), one finds the relevant quantities Q^i:

$$\begin{aligned} Q^1(r,\Psi_i,\dot{\Psi}_i) &= \frac{1}{2}\left(\frac{2}{r}\dot{\Psi}_1 + (2m\frac{\alpha+Er}{r} - \frac{2\nu^2}{r^2})\Psi_1 - \nu\frac{2r+\Gamma}{r^3}\Psi_2\right), \\ Q^2(r,\Psi_i,\dot{\Psi}_i) &= \frac{1}{2}\left(\frac{2}{r}\dot{\Psi}_2 + (2m\frac{\alpha+Er}{r} - \frac{2\nu^2}{r^2} - \frac{2}{r^2} - \frac{4\Gamma}{r^3} - \frac{\Gamma^2}{r^4})\Psi_2 - 2\nu\frac{2r+\Gamma}{r^3}\Psi_1\right). \end{aligned} \tag{56}$$

By direct calculation, according to the formulas (54), for the first invariant ε^i we find two invariants:

$$\begin{aligned} \varepsilon^1 &= \frac{\nu\Psi_2(\Gamma+2r)}{r^3} + \Psi_1(-2mE + \frac{2\nu^2}{r^2} - \frac{2m\alpha}{r}) - \frac{\dot{\Psi}_1}{r}, \\ \varepsilon^2 &= \frac{2\nu\Psi_1(\Gamma+2r)}{r^3} + \Psi_2(-2mE + \frac{\Gamma^2}{r^4} + \frac{4\Gamma}{r^3} + \frac{2(\nu^2+1)}{r^2} - \frac{2m\alpha}{r}) - \frac{\dot{\Psi}_2}{r}; \end{aligned} \tag{57}$$

$$P^i_{\ j} = \begin{vmatrix} 2m\frac{\alpha+Er}{r} - \frac{2\nu^2}{r^2} & -\frac{(2r+\Gamma)\nu}{r^3} \\ -\frac{2(2r+\Gamma)\nu}{r^3} & -\frac{\Gamma^2}{r^4} - \frac{4\Gamma}{r^3} + 2m\frac{\alpha+Er}{r} - \frac{2(\nu^2+1)}{r^2} \end{vmatrix}. \tag{58}$$

The eigenvalues Λ_1, Λ_2 of the second invariant are given by the formula

$$\begin{aligned} \Lambda_{1,2} = 2mE &+ \frac{1-2\nu^2}{r^2} + \frac{2m\alpha}{r} - \frac{(\Gamma+2r)^2}{2r^4} \pm \\ &\pm \frac{\sqrt{\left(\Gamma^2+2r^2+4\Gamma r\right)^2 + 8\nu^2 r^2(\Gamma+2r)^2}}{2r^4}, \end{aligned} \tag{59}$$

and the typical behavior of eigenvalues at different j is presented in figure 1.

Let us specify their behavior near the singular points $r=0$, $r=\infty$, and $r=-\Gamma/2$:

$$\begin{aligned} &r \to 0,\ \Lambda_1 \to \frac{2m\alpha}{r} > 0,\ \Lambda_2 \to -\frac{\Gamma^2}{r^4} < 0; \qquad r \to \infty,\ \Lambda_1, \Lambda_2 \to 2mE < 0; \\ &r \to -\frac{\Gamma}{2},\ \Lambda_1 \to 2mE - \frac{8\nu^2}{\Gamma^2} - \frac{4m\alpha}{\Gamma} < 0, \quad \Lambda_2 \to 2mE - \frac{8\left(\nu^2-1\right)}{\Gamma^2} - \frac{4m\alpha}{\Gamma} < 0. \end{aligned} \tag{60}$$

The behavior of the real parts of eigenvalues near the singular points $r = 0, \infty, -\Gamma/2$ correlates with the properties of solutions near the points $r = 0, \infty, -\Gamma/2$ for quantum mechanical bound states.

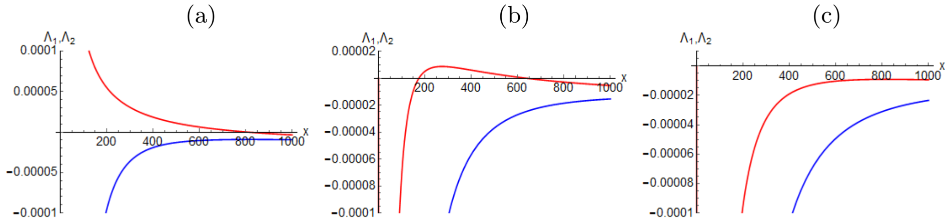

FIG. 1: The dependencies of eigenvalues Λ_1 (red) and Λ_1 (blue) on radial coordinate ($x = mr$) at different j: (a) $j = 1$, (b) $j = 2$, (c) $j = 3$. We used following parameters: $\Gamma m = 1$, $E/m = -0.000009$.

The next step is to construct a Lagrangian function L for the phase space $\dot{\Psi}_i$, Ψ_i, defined by (56). We will search for the function L in the form

$$L = g_{ij}(r) y^i y^j + b_j(r,x) y^j, \qquad b_j(r,x) = h_{ij}(r)\, x^i. \tag{61}$$

By substituting (61) into (53) and assuming that the tensor g_{ij} is diagonal, $g_{12} = g_{21} = 0$, one gets

$$\begin{aligned} Q^1 &= \frac{2\dot{g}_{11} y^1 + \dot{h}_{21} x^2 + \dot{h}_{11} x^1 + (h_{21} - h_{12})\, y^2}{4 g_{11}}, \\ Q^2 &= \frac{2\dot{g}_{22} y^2 + \dot{h}_{22} x^2 + \dot{h}_{12} x^1 + (h_{12} - h_{21})\, y^1}{4 g_{22}}. \end{aligned} \tag{62}$$

After equating particular terms in (56) with the corresponding terms in (62) (remembering also that $x^i = \Psi_i$; $y^i = \dot{\Psi}_i$), we derive the system of equations with respect to $g_{ij}(r)$ and $h_{ij}(r)$:

$$h_{21} - h_{12} = 0, \qquad \frac{\dot{g}_{11}}{2g_{11}} = \frac{1}{r}, \qquad \frac{\dot{g}_{22}}{2g_{22}} = \frac{1}{r},$$

$$\frac{\dot{h}_{11}}{4g_{11}} = \frac{2mr(\alpha + Er) - 2\nu^2}{2r^2}, \qquad \frac{\dot{h}_{21}}{4g_{11}} = -\frac{\nu(\Gamma + 2r)}{2r^3},$$

$$\frac{\dot{h}_{12}}{4g_{22}} = -\frac{\nu(\Gamma + 2r)}{r^3}, \qquad \frac{\dot{h}_{22}}{4g_{22}} = -\frac{\Gamma^2}{2r^4} - \frac{r\left(\nu^2 - mr(Er + \alpha) + 1\right) + 2\Gamma}{r^3}.$$

Its solution is given by the relations:

$$\begin{aligned} g_{11} = 2C_1 r^2,\ g_{22} = C_1 r^2,\ h_{12} = h_{21} &= C_2 - 4\nu C_1\,(\Gamma \ln r + 2r), \\ h_{11} &= C_2 - 4C_1\,(\frac{-2mEr^3}{3} - m\alpha r^2 + 2\nu^2 r), \\ h_{22} = C_2 - 2C_1 \left[-\frac{2mEr^3}{3} - m\alpha r^2 - \frac{\Gamma^2}{r} \right. &\left. + 4\Gamma \ln r + 2(\nu^2 + 1) r\right], \end{aligned} \tag{63}$$

where C_1, C_2 are the arbitrary constants. Finally, the Lagrangian L can be written in the form

$$\begin{aligned} L = C_1 \big[r^2\, (2\dot{\Psi}_1^2 + \dot{\Psi}_2^2) - 4(-\frac{2}{3} mEr^3 - m\alpha r^2 + 2\nu^2 r)\, \dot{\Psi}_1 \Psi_1 - \\ -2\,[-\frac{2mEr^3}{3} - m\alpha r^2 - \frac{\Gamma^2}{r} + 4\Gamma \ln r + 2(\nu^2 + 1) r\,] \dot{\Psi}_2 \Psi_2 - \\ -4\nu\, (\Gamma \ln r + 2r)\, (\dot{\Psi}_1 \Psi_2 + \dot{\Psi}_2 \Psi_1) \big] + C_2 [\Psi_1 \dot{\Psi}_1 + \Psi_2 \dot{\Psi}_2] + C_3. \end{aligned} \tag{64}$$

It can be readily checked that the last summands with C_2 and C_3 give zero at the substitution into the formula (53) for the coefficients Q^i and, consequently, this does not change the equation of motion. So, the Lagrangian has the arbitrariness up to the term $C_2[\Psi_1 \dot{\Psi}_1 + \Psi_2 \dot{\Psi}_2] + C_3$.

The task of construction of the Lagrangian function may be solved even without the restriction $b_j(r,x) = h_{ij}(r)\, x^i$. In this case, substituting (61) into (53) and assuming that the tensor g_{ij} is diagonal ($g_{12} = g_{21} = 0$), we derive

$$Q^1 = \frac{1}{4g_{11}}\left(2\dot g_{11} y^1 + \frac{\partial b_1}{\partial r} + (\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1})y^2\right),$$
$$Q^2 = \frac{1}{4g_{22}}\left(2\dot g_{22} y^2 + \frac{\partial b_2}{\partial r} + (\frac{\partial b_2}{\partial x^1} - \frac{\partial b_1}{\partial x^2})y^2\right).$$

Equating the terms from (56) to the corresponding terms from (65), we obtain the system of equations with respect to $g_{ij}(r)$ and $b_j(r,x)$:

$$\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1} = 0, \qquad \frac{\dot g_{11}}{2g_{11}} = \frac{1}{r}, \qquad \frac{\dot g_{22}}{2g_{22}} = \frac{1}{r},$$

$$\frac{1}{4g_1}\frac{\partial b_1}{\partial r} = \frac{x^1\left(r(2m\alpha + 2mEr) - 2\nu^2\right)}{2r^2} - \frac{\nu x^2(\Gamma + 2r)}{2r^3},$$

$$\frac{1}{4g_2}\frac{\partial b_2}{\partial r} = -\frac{\nu x^1(\Gamma + 2r)}{r^3} - \frac{x^2\left(\Gamma^2 + r^2\left(2\nu^2 - 2mr(Er + \alpha) + 2\right) + 4\Gamma r\right)}{2r^4}.$$

Its solution is given by the formulas:

$$g_{11} = 2C_1 r^2, \qquad g_{22} = C_1 r^2,$$

$$b_1 = B_1(x^1,x^2) - 4C_1\left\{r x^1[2\nu^2 - \frac{2mEr^2}{3} - mr\alpha] + \nu x^2(\Gamma \ln r + 2r)\right\},$$

$$b_2 = B_2(x^1,x^2) - 4C_1\{x^2[\,-\frac{\Gamma^2}{2r} + r\left(\nu^2+1\right) - \frac{mEr^3}{3} - \frac{m\alpha r^2}{2}$$

$$+2\Gamma\ln r\,] + \nu x^1(\Gamma\ln r + 2r)\}\,,$$

where C_1 is an arbitrary constant.

The two functions $B_1(x^1,x^2)$ and $B_2(x^1,x^2)$ obey the following restriction

$$\frac{\partial B_1(x^1,x^2)}{\partial x^2} - \frac{\partial B_2(x^1,x^2)}{\partial x^1} = 0. \tag{65}$$

From (65) we conclude that the 2-dimensional vector field B_1, B_2 can be represented as a gradient of a scalar function

$$B_1(x^1,x^2) = \frac{\partial}{\partial x^1}\varphi(x^1,x^2), \quad B_2(x^1,x^2) = \frac{\partial}{\partial x^2}\varphi(x^1,x^2), \quad B_i = \mathrm{grad}\,\varphi\,. \tag{66}$$

Therefore, there exist some freedom in choosing the Lagrangian (the constant C_1 may be taken as 1):

$$L = 2r^2(y^1)^2 + r^2(y^2)^2 + \left(\frac{2\Gamma^2}{r} + \frac{4}{3}mEr^3 + 2m\alpha r^2 - 4\left(\nu^2+1\right)r - 8\Gamma\ln r\right)x^2y^2 -$$

$$-4\nu(\Gamma\ln r + 2r)x^1y^2 - 4\nu(\Gamma\ln r + 2r)x^2y^1 - r(8\nu^2 - \frac{8mEr^2}{3} - 4m\alpha r)x^1y^1 +$$

$$+y^1\frac{\partial\varphi}{\partial x^1} + y^2\frac{\partial\varphi}{\partial x^2}.$$

VII. DISCUSSION

We have emerged from the relativistic Duffin-Kemmer-Petiau equation for spin 1 particle with anomalous magnetic moment in presence of an external Coulomb field. By diagonalizing the operators of energy, square and third projection of the total angular momentum, we have derived the system of 10 radial equations. After diagonalizing the spatial reflection operator, we have split the system into systems of 4 and 6 equations for states with parities $P = (-1)^{j+1}$ and $P = (-1)^{j}$, respectively.

The interaction terms due to anomalous magnetic moment are present only in the system referring to states with parity $P = (-1)^{j}$. In order to simplify the problem, a transition to the non-relativistic approximation has been performed.

The states with minimal $j = 0$ are simplest, being described by a 2-nd order equation of the double confluent Heun type. By imposing the known transcendency condition, we have derived some quantization rule, which leads to the energy spectrum of the structure $E = -\mathrm{const}/n^2$. The numerical values for energies seem to be physically interpretable, though they do not depend on the anomalous magnetic moment.

For states with $j = 1, 2, ...$, after performing the non-relativistic approximation we have obtained a system of two 2-nd order linked differential equations for radial functions. This system gives a 4-th order equation for one function. Its Frobenius solutions were constructed, and the convergence of the involved power series with 8- and 9-terms recurrence relations, were studied.

All the constructed solutions are exact, but they are formal because no correct quantization rules for energies of bound states are known yet.

To study the quantum-mechanical problem of spin 1 particle with anomalous magnetic moment in the external Coulomb field we have additionally applied the known geometrical method based on the use of KCC-invariants. The first and the second invariants have been calculated. It has been shown that the different branches of the solutions converge near the singular points $\infty, -\Gamma/2$, and may either converge or diverge near the singular points $r = 0$. This correlates with the expected behavior of solutions for bound states. The explicit Lagrangians referring to the geometrical problem were found. It was shown that Lagrangians have the arbitrariness up to some special terms, which may be considered as a specific gauge freedom.

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[19] we further use the established tetrad formalism, e.g., see [13].