A.D. Koralkov, A.P. Safronov, E.M. Ovsiyuk

Dirac particle in the Coulomb field on the background of hyperbolic Lobachevsky and spherical Riemann models

The known systems of radial equations describing relativistic hydrogen atom on the base of Dirac equation in Lobachevsky – Riemann spaces models of constant curvature are investigated. In both geometrical models differential equations second order with six regular singular points are found, their exact solutions of Frobenius type are constructed. For producing quantization rule for energy values we use a condition separating transcendental Frobenius solutions. This provides us with energy spectra which are physically interpretable and are similar to spectra arising from scalar Klein – Fock – Gordon equation in these space two models. These spectra are similar to those previously arising from studying the same radial systems of equations when applying semi-classical approximation.

Key words: Dirac particle, Coulomb field, Lobachevsky space, Frobenius solutions, transcendency conditions, energy spectrum

1 Hydrogen atom in Lobachevsky space

In Lobachevsky space H_3, the Dirac equation in presence of Coulomb field (see [1–19]) after separation the variables leads to a radial system of two equations (we adhere notations from [19, 20]; radial variable becomes is dimensionless by dividing on the curvature radius ρ, $r \in (0, \infty)$)

$$(\frac{d}{dr} + \frac{\nu}{\sinh r})f + (E + \frac{e}{\tanh r} + m)g = 0, \quad (\frac{d}{dr} - \frac{\nu}{\sinh r})g - (E + \frac{e}{\tanh r} - m)f = 0. \tag{1.1}$$

After transforming the system (1.1) to the variable $\tanh \frac{r}{2} = z, z \in (0, 1)$, we obtain

$$\frac{d}{dz}f + \frac{\nu}{z}f + \left[\frac{e}{z} + \frac{-E-e-m}{z-1} + \frac{E-e+m}{z+1}\right] g = 0\,,$$
$$\frac{d}{dz}g - \frac{\nu}{z}g - \left[\frac{e}{z} + \frac{-E-e+m}{z-1} + \frac{E-e-m}{z+1}\right] f = 0\,. \tag{1.2}$$

Whence it follows a 2-nd order equation for $f(z)$:

$$\frac{d^2 f}{dz^2} + \left[\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z+1} - 2\,\frac{ez+E+m}{ez^2+2(E+m)\,z+e}\right]\frac{df}{dz}$$
$$+\left[2\,\frac{2Ee^2-(E+m)\,\nu}{ez} + \frac{-(E+e)^2+m^2+\nu}{z-1} + \frac{(E-e)^2-m^2-\nu}{z+1} + \frac{e^2-\nu^2}{z^2}\right.$$
$$\left.+\frac{(E+e)^2-m^2}{(z-1)^2} + \frac{(E-e)^2-m^2}{(z+1)^2} + \frac{2\,\nu\,[ez(E+m)+2\,(E+m)^2-e^2]}{e\,[\,ez^2+2(E+m)\,z+e\,]}\right] f = 0\,. \tag{1.3}$$

The lasr equation has 6 singular points (let $\frac{E+m}{e} = \sigma > 0$):

$$0\,,\ \ \infty\,,\ \ \pm 1,\ \ z_{1,\,2} = -\sigma \pm \sqrt{\sigma^2-1} \quad (z_1 z_2 = 1\,,\ \ z_1+z_2 = -2\sigma)\,. \tag{1.4}$$

Eq. (1.3) may be re-written differently

$$\frac{d^2 f}{dz^2} + \left[\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{z-z_1} - \frac{1}{z-z_2}\right]\frac{df}{dz}$$

$$+\left[\frac{4Ee-2\sigma\,\nu}{z}-\frac{(E+e)^2-m^2-\nu}{z-1}+\frac{(E-e)^2-m^2-\nu}{z+1}+\frac{e^2-\nu^2}{z^2}\right.$$
$$\left.+\frac{(E+e)^2-m^2}{(z-1)^2}+\frac{(E-e)^2-m^2}{(z+1)^2}+\frac{A}{z-z_1}+\frac{B}{z-z_2}\right]f=0\,, \tag{1.5}$$

where

$$2\nu\frac{\sigma\,z+2\,\sigma^2-1}{(z-z_1)(z-z_2)}=\frac{A}{z-z_1}+\frac{B}{z-z_2},\quad A=2\nu\frac{\sigma z_1+2\sigma^2-1}{z_1-z_2},\quad B=2\nu\frac{\sigma z_2+2\sigma^2-1}{z_2-z_1}.$$

For shortness let us introduce notations

$$C=(E+e)^2-m^2,\quad D=(E-e)^2-m^2,\quad 4Ee=C-D,$$

then eq. (1.5) reads

$$\frac{d^2f}{dz^2}+\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2}\right)\frac{df}{dz}+ \tag{1.6}$$
$$+\left(\frac{C-D-2\sigma\nu}{z}-\frac{C-\nu}{z-1}+\frac{D-\nu}{z+1}+\frac{e^2-\nu^2}{z^2}+\frac{C}{(z-1)^2}+\frac{D}{(z+1)^2}+\frac{A}{z-z_1}+\frac{B}{z-z_2}\right)f=0.$$

Near the points $r=0,+1,-1$, solutions behave as

$$f\sim(z-1)^{\alpha},\ \alpha=\pm\sqrt{-C}\,;\quad f\sim(z+1)^{\beta},\ \beta=\pm\sqrt{-D}\,;\quad f\sim z^M,\ M=\pm\sqrt{\nu^2-e^2}\,. \tag{1.7}$$

Let us search Frobenius type solutions in the form

$$f(z)=x^M(z-1)^{\alpha}(z+1)^{\beta}\varphi(z)\,; \tag{1.8}$$

for function $\varphi(z)$ we derive the following equation

$$\frac{d^2\varphi}{dz^2}+\left[\frac{2M+1}{z}+\frac{2\alpha+1}{z-1}+\frac{2\beta+1}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2}\right]\frac{d\varphi}{dz}+$$
$$+\left[\frac{M^2+e^2-\nu^2}{z^2}+\frac{\alpha^2+C}{(z-1)^2}+\frac{\beta^2+D}{(z+1)^2}+\right.$$
$$+\frac{C-D-(\alpha-\beta)\,(2M+1)-2\sigma(\nu+M)}{z}+\frac{M\,(z_1+2\,\sigma\,z_1\,z_2+z_2)}{z\,z_2\,z_1}+$$
$$+\frac{M+\alpha/2+\beta/2-C+\nu+2M\alpha+\alpha\beta}{z-1}-\frac{\alpha\,(1-z_1\,z_2)}{(z-1)\,(z_1-1)\,(z_2-1)}-$$
$$-\frac{M+\alpha/2+\beta/2-D+\nu+2M\beta+\alpha\beta}{z+1}+\frac{\beta\,(1-z_1\,z_2)}{(z+1)\,(z_1+1)\,(z_2+1)}+$$
$$\left.+\frac{1}{z-z_1}\left(A-\frac{\alpha}{z_1-1}-\frac{\beta}{z_1+1}-\frac{M}{z_1}\right)+\frac{1}{z-z_2}\left(B-\frac{\alpha}{z_2-1}-\frac{\beta}{z_2+1}-\frac{M}{z_2}\right)\right]\varphi=0\,.$$

Let us impose restriction on parameters:

$$M=\pm\sqrt{\nu^2-e^2},\quad \alpha=\pm\sqrt{-C}=\pm\sqrt{m^2-(E+e)^2},\quad \beta=\pm\sqrt{-D}=\pm\sqrt{m^2-(E-e)^2};$$

it should be emphasized that to bound states may correspond the following values

$$M=+\sqrt{\nu^2-e^2}\,,\quad \alpha=+\sqrt{m^2-(E+e)^2}\,,\qquad \beta=\pm\sqrt{m^2-(E-e)^2}\,. \tag{1.9}$$

With (1.9) in mind, for function $\varphi(z)$ we obtain the equation

$$\frac{d^2\varphi}{dz^2}+\left[\frac{2M+1}{z}+\frac{2\alpha+1}{z-1}+\frac{2\beta+1}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2}\right]\frac{d\varphi}{dz}+$$

$$+\left[\frac{C-D-(\alpha-\beta)(2M+1)-2\sigma(\nu+M)}{z}+\frac{M\,(z_1+2\,\sigma\,z_1\,z_2+z_2)}{z\,z_2\,z_1}+\right.$$

$$+\frac{M+\alpha/2+\beta/2-C+\nu+2M\alpha+\alpha\beta}{z-1}-\frac{\alpha\,(1-z_1\,z_2)}{(z-1)\,(z_1-1)\,(z_2-1)}-$$

$$-\frac{M+\alpha/2+\beta/2-D+\nu+2M\beta+\alpha\beta}{z+1}+\frac{\beta\,(1-z_1\,z_2)}{(z+1)\,(z_1+1)\,(z_2+1)}+$$

$$\left.+\frac{1}{z-z_1}\left(A-\frac{\alpha}{z_1-1}-\frac{\beta}{z_1+1}-\frac{M}{z_1}\right)+\frac{1}{z-z_2}\left(B-\frac{\alpha}{z_2-1}-\frac{\beta}{z_2+1}-\frac{M}{z_2}\right)\right]\varphi=0\,.$$

It is convenient to use its shortening presentation

$$\frac{d^2\varphi}{dz^2}+(\frac{P_1}{z}+\frac{P_2}{z-1}+\frac{P_3}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2})\frac{d\varphi}{dz}+$$
$$+(\frac{Q_1}{z}+\frac{Q_2}{z-1}+\frac{Q_3}{z+1}+\frac{Q_4}{z-z_1}+\frac{Q_5}{z-z_2})\varphi=0\,.$$

Multiplying the last equation by $z(z-1)(z+1)(z-z_1)(z-z_2)$ we get

$$\left[z^5+(-z_1-z_2)\,z^4+(z_1\,z_2-1)\,z^3+(z_1+z_2)\,z^2-z_1\,z_2\,z\right]\frac{d^2\varphi}{dz^2}+$$

$$+\left[(P_1+P_2+P_3-2)\,z^4+\{(1-P_1-P_2-P_3)\,z_1+(1-P_1-P_2-P_3)\,z_2+P_2-P_3\}\,z^3+\right.$$

$$+(2-P_1-P_2\,z_1+P_3\,z_2+P_2\,z_1\,z_2+P_1\,z_1\,z_2+P_3\,z_1\,z_2+P_3\,z_1-P_2\,z_2)\,z^2+$$

$$\left.+(-z_1-z_2+P_1\,z_1-P_3\,z_1\,z_2+P_1\,z_2+P_2\,z_1\,z_2)\,z-P_1\,z_1\,z_2\right]\frac{d\varphi}{dz}+$$

$$+[(Q_1+Q_2+Q_3+Q_4+Q_5)z^4+\{(-Q_1-Q_2-Q_3-Q_5)z_1+$$

$$+(-Q_1-Q_2-Q_3-Q_4)z_2+Q_2-Q_3\}z^3+$$

$$+(Q_3\,z_1\,z_2+Q_2\,z_1\,z_2+Q_3\,z_2-Q_1-Q_4-Q_5+Q_1\,z_1\,z_2+Q_3\,z_1-Q_2\,z_2-Q_2\,z_1)\,z^2+$$

$$+(Q_1\,z_2+Q_2\,z_1\,z_2+Q_5\,z_1+Q_1\,z_1-Q_3\,z_1\,z_2+Q_4\,z_2)\,z-Q_1\,z_1\,z_2]\,\varphi=0\,.$$

Solutions $\varphi(z)$ may be constructed in the form of power series, $\varphi=\sum_{n=0}^{\infty}d_n z^n$; after performing needed calculation we derive 6-term recurrent relations:

$$k\geq 4,\qquad (Q_1+Q_2+Q_3+Q_4+Q_5)\;d_{k-4}+$$

$$+[(k-3)(k-4)+(P_1+P_2+P_3-2)\;(k-3)+$$

$$+(-Q_1-Q_2-Q_3-Q_5)\,z_1+(-Q_1-Q_2-Q_3-Q_4)\,z_2+Q_2-Q_3]\;d_{k-3}+$$

$$+[(-z_1-z_2)\;(k-2)(k-3)+\{(1-P_1-P_2-P_3)\,z_1+(1-P_1-P_2-P_3)\,z_2+P_2-P_3\}\;(k-2)+$$

$$+Q_3z_1z_2+Q_2z_1z_2+Q_3z_2-Q_1-Q_4-Q_5+Q_1z_1z_2+Q_3\,z_1-Q_2z_2-Q_2z_1]\,d_{k-2}+$$

$$+[(z_1z_2-1)\,(k-1)(k-2)+(2-P_1-P_2z_1+P_3z_2+P_2z_1z_2+P_1z_1z_2+P_3z_1z_2+P_3z_1-P_2z_2)\,(k-1)+$$

$$+Q_1\,z_2+Q_2\,z_1\,z_2+Q_5\,z_1+Q_1\,z_1-Q_3\,z_1\,z_2+Q_4\,z_2]\;d_{k-1}+$$

$$+[(z_1+z_2)\;k\,(k-1)+(-z_1-z_2+P_1\,z_1-P_3\,z_1\,z_2+P_1\,z_2+P_2\,z_1\,z_2)\,k-Q_1\,z_1\,z_2]\;d_k+$$

$$+\left[-z_1\, z_2\,(k+1)\,k - P_1\, z_1\, z_2\,(k+1)\right]\, d_{k+1} = 0\,. \qquad (1.10)$$

To analyze convergence of power series, we apply Poincaré – Perrone method, so divide relation (1.10) by . $k^2 d_{k-4}$

$$
\begin{gathered}
(Q_1+Q_2+Q_3+Q_4+Q_5)+\\
+\left[(k-3)(k-4)+(P_1+P_2+P_3-2)\;(k-3)+\right.\\
\left.+\,(-Q_1-Q_2-Q_3-Q_5)\,z_1+(-Q_1-Q_2-Q_3-Q_4)\,z_2+Q_2-Q_3\right]\,\frac{d_{k-3}}{d_{k-4}}+\\
+\left[(-z_1-z_2)\;(k-2)(k-3)+\{(1-P_1-P_2-P_3)\,z_1+(1-P_1-P_2-P_3)\,z_2+P_2-P_3\}\;(k-2)+\right.\\
\left.+\,Q_3z_1z_2+Q_2z_1z_2+Q_3z_2-Q_1-Q_4-Q_5+Q_1z_1z_2+Q_3z_1-Q_2z_2-Q_2z_1\right]\frac{d_{k-2}}{d_{k-3}}\frac{d_{k-3}}{d_{k-4}}+\\
+\left[(z_1z_2-1)\,(k-1)(k-2)+(2-P_1-P_2z_1+P_3z_2+P_2z_1z_2+P_1z_1z_2+P_3z_1z_2+P_3z_1-P_2z_2)\,(k-1)+\right.\\
\left.+Q_1\,z_2+Q_2\,z_1\,z_2+Q_5\,z_1+Q_1\,z_1-Q_3\,z_1\,z_2+Q_4\,z_2\right]\,\frac{d_{k-1}}{d_{k-2}}\frac{d_{k-2}}{d_{k-3}}\frac{d_{k-3}}{d_{k-4}}+\\
+\left[(z_1+z_2)\,k(k-1)+(-z_1-z_2+P_1z_1-P_3z_1z_2+P_1z_2+P_2z_1z_2)\,k-Q_1z_1z_2\right]\,\frac{d_k}{d_{k-1}}\frac{d_{k-1}}{d_{k-2}}\frac{d_{k-2}}{d_{k-3}}\frac{d_{k-3}}{d_{k-4}}+\\
+\left[-z_1z_2(k+1)k-P_1\,z_1\,z_2\,(k+1)\right]\,\frac{d_{k+1}}{d_k}\frac{d_k}{d_{k-1}}\frac{d_{k-1}}{d_{k-2}}\frac{d_{k-2}}{d_{k-3}}\frac{d_{k-3}}{d_{k-4}}=0\,.
\end{gathered}
$$

, and tend $k \to \infty$. In this way, for quantity a $R = \lim_{k\to\infty}(d_{k-3}/d_{k-4})$ we derive an algebraic equation

$$R - (z_1+z_2)\,R^2 + (z_1z_2-1)\,R^3 + (z_1+z_2)\,R^4 - z_1z_2R^5 = 0 \quad\Longrightarrow\quad R = 0, \pm 1, \frac{1}{z_1}, \frac{1}{z_2}.$$

Therefore< possible convergence radii are

$$R_{\text{conv}} = \left|\frac{1}{R}\right| = +1\,,\ +\infty\,,\ |z_1|\,,\ |z_2|. \qquad (1.16)$$

Turning to recurrent formulas (1.10), we can see that coefficient at d_{k-4} vanish identically: $Q_1 + Q_2 + Q_3 + Q_4 + Q_5 = 0,$. This means that in (1.10) actually we have 5-term recurrent relation

$$S_{k-3}d_{k-3} + S_{k-2}d_{k-2} + S_{k-1}d_{k-1} + S_k d_k + S_{k+1}d_{k+1} = 0\,. \qquad (1.11)$$

As a quantization rule, let us apply transcendency condition for Frobenius type functions, this yields

$$
\begin{gathered}
S_{k-3}=0,\quad k\geq 3,\qquad (k-3)(k-4)+(P_1+P_2+P_3-2)\;(k-3)+\\
+\,(-Q_1-Q_2-Q_3-Q_5)\,z_1+(-Q_1-Q_2-Q_3-Q_4)\,z_2+Q_2-Q_3=0\,, \qquad (1.18)
\end{gathered}
$$

with the use of the above formulas for coefficients it reads

$$
\begin{gathered}
k^2+(2\,M+2\,\alpha+2\,\beta-6)\,k-(B-2\,\sigma\,\nu)\,z_1-(A-2\,\sigma\,\nu)\,z_2+\\
+\,(2\,M+2\,\beta-6)\,\alpha+(2\,M-6)\,\beta-6\,M-C-D+2\,\nu+9=0\,.
\end{gathered}
$$

Whence substituting expressions for A, B, C, D, we derive

$$
\begin{gathered}
k\geq 3,\qquad k^2+2\;(M+\alpha+\beta-3)\,k+2\,\sigma\,\nu\;(z_1+z_2)\,+\\
+2\;(M+\beta-3)\,\alpha+2\;(M-3)\,\beta+9-6\,M+2\,m^2+4\,\sigma^2\nu-2\,e^2-2\,E^2=0\,. \qquad (1.12)
\end{gathered}
$$

Now we allow for the formulas

$$\alpha = +\sqrt{m^2-(E+e)^2}, \quad \beta = \pm\sqrt{m^2-(E-e)^2},$$

$$M = \sqrt{\nu^2-e^2}, \quad z_1 z_2 = 1, \quad z_1 + z_2 = -2\sigma = -2\frac{E+m}{e} \;;$$

then (1.12) take the form (it depends on the choice for β):

$\beta = +\sqrt{m^2-(E-e)^2}$,

$$\left(\sqrt{m^2-(E+e)^2} + \sqrt{m^2-(E-e)^2} + k - 3 + \sqrt{\nu^2-e^2}\right)^2 - (\nu^2-e^2) = 0\,; \qquad (1.13)$$

$\beta = -\sqrt{m^2-(E-e)^2}$,

$$\left(\sqrt{m^2-(E+e)^2} - \sqrt{m^2-(E-e)^2} + k - 3 + \sqrt{\nu^2-e^2}\right)^2 - (\nu^2-e^2) = 0\,. \qquad (1.14)$$

We can follow both variants ($\pm$ sign). let us factorize expressions in product of two ones (let $k-3=n, n=0,1,...$):

$$\left(\sqrt{m^2-(E+e)^2} \pm \sqrt{m^2-(E-e)^2} + n + \sqrt{\nu^2-e^2} - \sqrt{\nu^2-e^2}\right) \times$$
$$\times \left(\sqrt{m^2-(E+e)^2} \pm \sqrt{m^2-(E-e)^2} + n + \sqrt{\nu^2-e^2} + \sqrt{\nu^2-e^2}\right) = 0,$$

that is

$$\left(\sqrt{m^2-(E+e)^2} \pm \sqrt{m^2-(E-e)^2} + n\right) \times$$
$$\times \left(\sqrt{m^2-(E+e)^2} \pm \sqrt{m^2-(E-e)^2} + n + \sqrt{\nu^2-e^2} + \sqrt{\nu^2-e^2}\right) = 0\,. \qquad (1.15)$$

For upper sign (when $\beta > 0$), the first multiplier is positive and cannot be equal to zero; therefore it remain only the following equation

$$\beta > 0, \quad \left(\sqrt{m^2-(E+e)^2} + \sqrt{m^2-(E-e)^2} + n + \sqrt{\nu^2-e^2} + \sqrt{\nu^2-e^2}\right) = 0\;; \qquad (1.16)$$

however it does not have any physical solutions because all term are positive.

For lower sign (when $\beta < 0$), we have an equation

$$\left(\sqrt{m^2-(E+e)^2} - \sqrt{m^2-(E-e)^2} + n\right) \times$$
$$\times \left(\sqrt{m^2-(E+e)^2} - \sqrt{m^2-(E-e)^2} + n + \sqrt{\nu^2-e^2} + \sqrt{\nu^2-e^2}\right) = 0$$

Alternatively, there arise two possibilities:

$$\sqrt{m^2-(E+e)^2} - \sqrt{m^2-(E-e)^2} + n = 0\,, \qquad (1.17)$$

and

$$\sqrt{m^2-(E+e)^2} - \sqrt{m^2-(E-e)^2} + n + \sqrt{\nu^2-e^2} + \sqrt{\nu^2-e^2} = 0\,. \qquad (1.18)$$

Eq. (1.17) does not contain the angular parameter $\nu = j + 1/2$, and it is no physical interest. The most e promising is eq. (1.18):

$$\sqrt{m^2-E^2-e^2+2eE} - \sqrt{m^2-E^2-e^2-2eE} = n + 2\sqrt{\nu^2-e^2} = 2N > 0\,. \qquad (1.19)$$

This equation gives

$$m^2 - E^2 - e^2 + 2eE = m^2 - E^2 - e^2 - 2eE + 4N\sqrt{m^2 - E^2 - e^2 - 2eE} + 4N^2,$$

that is

$$eE - N^2 = +N\sqrt{m^2 - E^2 - e^2 - 2eE}.$$

and further

$$E^2(e^2 + N^2) = N^2(m^2 - e^2) - N^4,$$

whence we arrive at the following formulas for energy spectrum

$$\frac{E}{m} = \sqrt{\frac{1 - (e^2 + N^2)/m^2}{1 + \frac{e^2}{N^2}}}\,, \quad N = \frac{n}{2} + \sqrt{\nu^2 - e^2}. \tag{1.20}$$

Expression under the square root in (1.20) must be positive

$$1 > \frac{e^2 + N^2}{m^2}. \tag{1.21}$$

This spectrum coincides with that found in [10, 14], when studying the same problem for Dirac equation in Lobachevsky space within WKB-approach.

The next question is does exist or not the possibility to get the same energy spectrum by imposing polynomial conditions. To this end we should turn to the recurrent formulas (1.11)

$$\underline{S_{k-3}}d_{k-3} + S_{k-2}\underline{d_{k-2}} + S_{k-1}\underline{d_{k-1}} + S_k\underline{d_k} + S_{k+1}d_{k+1} = 0$$

and for energies given by (1.20) check the values of three coefficients

$$d_{k-2} = 0, \quad d_{k-1} = 0, \quad d_k = 0\,. \tag{1.22}$$

If the equalities (1.22) are valid then from recurrent formula it follows that series becomes a polynomial:

$$d_{k+1} = 0, \quad d_{k+2} = 0, \quad d_{k+3} = 0... \,.$$

Numerical study shows that equalities of the type (1.22) are not true.

2 Hydrogen atom is spherical Riemann space

In spherical space S_3, we have the system of differential equations [19]:

$$(\frac{d}{dr} + \frac{\nu}{\sin r})f + (E + \frac{e}{\tan r} + m)g = 0, \ (\frac{d}{dr} - \frac{\nu}{\sin r})g - (E + \frac{e}{\tan r} - m)f = 0; \tag{2.1}$$

dimensionless radial coordinate vary in the interval $r \in [0, \pi]$. In other variable

$$z = i\tan\frac{r}{2}, \quad \cos r = \frac{1+z^2}{1-z^2}, \quad \sin r = \frac{-2iz}{1-z^2}, \quad z \in [0, +i\infty); \tag{2.2}$$

the above system takes the form

$$\frac{df}{dz} + \frac{\nu}{z}f + \left(\frac{e}{z} + \frac{iE - e + im}{z-1} + \frac{-iE - e - im}{z+1}\right)g = 0\,,$$
$$\frac{dg}{dz} - \frac{\nu}{z}g + \left(-\frac{e}{z} + \frac{-iE + e + im}{z-1} + \frac{iE + e - im}{z+1}\right)f = 0\,. \tag{2.3}$$

whence it follows the 2-nd order equation for $f(z)$:

$$\frac{d^2 f}{dz^2} + \left[\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z+1} + 2\,\frac{-ez + iE + im}{ez^2 - 2i\,(E+m)\,z + e} \right] \frac{df}{dz} +$$

$$+ \left[-2i\,\frac{2\,Ee^2 - (E+m)\,\nu}{ez} + \frac{e^2-\nu^2}{z^2} + \frac{(E+ie)^2 - m^2 + \nu}{z-1} + \frac{-\,(E+ie)^2 + m^2}{(z-1)^2} + \right.$$

$$\left. + \frac{-\,(E-ie)^2 + m^2 - \nu}{z+1} + \frac{-\,(E-ie)^2 + m^2}{(z+1)^2} + \frac{2\nu\left[iez(E+m) + 2(E+m)^2 + e^2\right]}{e\left[-ez^2 + 2i\,(E+m)\,z - e\right]} \right] f = 0. \qquad (2.4)$$

Eq. (2.4) has 6 singular points (let $\frac{E+m}{e} = \sigma > 0$)

$$0\,,\ \infty\,,\ \pm 1,\ z_{1,\,2} = i\left(\sigma \pm \sqrt{\sigma^2+1}\right)\ ; \qquad (2.5)$$

physical region for variable z is the interval $z \in [0, +i\infty)$. Eq. (2.4) may be written differently

$$\frac{d^2 f}{dz^2} + \left[\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{z-z_1} - \frac{1}{z-z_2} \right] \frac{df}{dz} +$$

$$+ \left[\frac{-4\,iEe + 2\,i\sigma\,\nu}{z} + \frac{e^2-\nu^2}{z^2} + \frac{(E+ie)^2 - m^2 + \nu}{z-1} + \frac{-\,(E+ie)^2 + m^2}{(z-1)^2} + \right.$$

$$\left. + \frac{-\,(E-ie)^2 + m^2 - \nu}{z+1} + \frac{-\,(E-ie)^2 + m^2}{(z+1)^2} + \frac{A}{z-z_1} + \frac{B}{z-z_2} \right] f = 0\,, \qquad (2.6)$$

where

$$A = -\frac{2\nu\,\left(iz_1\,\sigma + 1 + 2\,\sigma^2\right)}{z_1 - z_2}, \quad B = -\frac{2\nu\,\left(iz_2\,\sigma + 1 + 2\,\sigma^2\right)}{z_2 - z_1}.$$

Below we use notations

$$C = -\,(E+ie)^2 + m^2, \quad D = -\,(E-ie)^2 + m^2, \quad -4iEe = C - D,$$

then eq. (2.6) takes the form

$$\frac{d^2 f}{dz^2} + \left[\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{z-z_1} - \frac{1}{z-z_2} \right] \frac{df}{dz} +$$

$$+ \left[\frac{C - D + 2\,i\sigma\,\nu}{z} + \frac{e^2-\nu^2}{z^2} - \frac{C-\nu}{z-1} + \frac{C}{(z-1)^2} + \frac{D-\nu}{z+1} + \frac{D}{(z+1)^2} + \frac{A}{z-z_1} + \frac{B}{z-z_2} \right] f = 0\,.$$

Frobenius solutions are searched in the form

$$f(z) = z^M (z-1)^\alpha (z+1)^\beta\ \varphi(z)\,; \qquad (2.7)$$

the function $\varphi(z)$ obeys equation

$$\frac{d^2\varphi}{dz^2} + \left[\frac{2M+1}{z} + \frac{2\alpha+1}{z-1} + \frac{2\beta+1}{z+1} - \frac{1}{z-z_1} - \frac{1}{z-z_2} \right] \frac{d\varphi}{dz} +$$

$$+ \left[\frac{M^2 + e^2 - \nu^2}{z^2} + \frac{\alpha^2 + C}{(z-1)^2} + \frac{\beta^2 + D}{(z+1)^2} + \right.$$

$$
+\frac{C-D-(\alpha-\beta)\,(2M+1)+2i\sigma(\nu+M)}{z}+\frac{M\,(z_1-2\,i\sigma\,z_1\,z_2+z_2)}{z\,z_2\,z_1}+
$$
$$
+\frac{M+\alpha/2+\beta/2-C+\nu+2M\alpha+\alpha\beta}{z-1}-\frac{\alpha\;(1-z_1\,z_2)}{(z-1)\,(z_1-1)\,(z_2-1)}-
$$
$$
-\frac{M+\alpha/2+\beta/2-D+\nu+2M\beta+\alpha\beta}{z+1}+\frac{\beta\;(1-z_1\,z_2)}{(z+1)\,(z_1+1)\,(z_2+1)}+
$$
$$
+\frac{1}{z-z_1}\;\left(A-\frac{\alpha}{z_1-1}-\frac{\beta}{z_1+1}-\frac{M}{z_1}\right)+\frac{1}{z-z_2}\;\left(B-\frac{\alpha}{z_2-1}-\frac{\beta}{z_2+1}-\frac{M}{z_2}\right)\Bigg]\;\varphi=0\;.
$$

Impose restrictions

$$
M=\pm\sqrt{\nu^2-e^2}\;,
$$

$$
\alpha=\pm\sqrt{-C}=\pm\sqrt{(E+ie)^2-m^2}=\pm\sqrt{E^2-m^2-e^2+2ieE}\;,
$$
$$
\beta=\pm\sqrt{-D}=\pm\sqrt{(E-ie)^2-m^2}=\pm\sqrt{E^2-m^2-e^2-2ieE}\,. \tag{2.8}
$$

To have solutions vanishing at the point $z=0$ ($r=0$), we must use positive value for M: $M=+\sqrt{\nu^2-e^2}$; near the point $z=+\infty$ ($r=\pi$) the multiplier before $\varphi(z)$ behaves as follows

$$
z^M(z-1)^\alpha(z+1)^\beta\sim x^{\sqrt{\nu^2-e^2}+(\alpha+\beta)}; \tag{2.9}
$$

depending on signs at α,β there exist 4 possibilities:

$$
\begin{array}{ll}
(-,-) & \alpha+\beta=-\sqrt{E^2-m^2-e^2+2ieE}-\sqrt{E^2-m^2-e^2-2ieE}<0\;;\\
(+,+) & \alpha+\beta=\sqrt{E^2-m^2-e^2+2ieE}+\sqrt{E^2-m^2-e^2-2ieE}>0\;;\\
(+,-) & \alpha+\beta=\sqrt{E^2-m^2-e^2+2ieE}-\sqrt{E^2-m^2-e^2-2ieE}\;\;\text{imaginary}\;;\\
(-,+) & \alpha+\beta=-\sqrt{E^2-m^2-e^2+2ieE}+\sqrt{E^2-m^2-e^2-2ieE}\;\;\text{imaginary}\;;
\end{array} \tag{2.10}
$$

only two first variants may give multiplier tending to zero, this requires the following inequality $M+\alpha+\beta<0$. The inequality $M+\alpha+\beta<0$ is definitely true for the case $(-,-)$.

Now we turn to equation for $\varphi(z)$:

$$
\frac{d^2\varphi}{dz^2}+\left[\frac{2M+1}{z}+\frac{2\alpha+1}{z-1}+\frac{2\beta+1}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2}\right]\,\frac{d\varphi}{dz}+
$$
$$
+\Bigg[\frac{C-D-(\alpha-\beta)\,(2M+1)+2i\sigma(\nu+M)}{z}+\frac{M\,(z_1-2\,i\sigma\,z_1\,z_2+z_2)}{z\,z_2\,z_1}+
$$
$$
+\frac{M+\alpha/2+\beta/2-C+\nu+2M\alpha+\alpha\beta}{z-1}-\frac{\alpha\;(1-z_1\,z_2)}{(z-1)\,(z_1-1)\,(z_2-1)}-
$$
$$
-\frac{M+\alpha/2+\beta/2-D+\nu+2M\beta+\alpha\beta}{z+1}+\frac{\beta\;(1-z_1\,z_2)}{(z+1)\,(z_1+1)\,(z_2+1)}+
$$
$$
+\frac{1}{z-z_1}\;\left(A-\frac{\alpha}{z_1-1}-\frac{\beta}{z_1+1}-\frac{M}{z_1}\right)+\frac{1}{z-z_2}\;\left(B-\frac{\alpha}{z_2-1}-\frac{\beta}{z_2+1}-\frac{M}{z_2}\right)\Bigg]\;\varphi=0\;.
$$

Re-write it shorter

$$
\frac{d^2\varphi}{dz^2}+(\frac{P_1}{z}+\frac{P_2}{z-1}+\frac{P_3}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2})\frac{d\varphi}{dz}+(\frac{Q_1}{z}+\frac{Q_2}{z-1}+\frac{Q_3}{z+1}+\frac{Q_4}{z-z_1}+\frac{Q_5}{z-z_2})\varphi=0\;.
$$

and by $z(z-1)(z+1)(z-z_1)(z-z_2)$:

$$\left[z^5 + (-z_1 - z_2)\, z^4 + (z_1\, z_2 - 1)\, z^3 + (z_1 + z_2)\, z^2 - z_1\, z_2\, z\right] \frac{d^2\varphi}{dz^2} +$$

$$+ \big[(P_1 + P_2 + P_3 - 2)\, z^4 + \{(1 - P_1 - P_2 - P_3)\, z_1 + (1 - P_1 - P_2 - P_3)\, z_2 + P_2 - P_3\}\, z^3 +$$

$$+ (2 - P_1 - P_2\, z_1 + P_3\, z_2 + P_2\, z_1\, z_2 + P_1\, z_1\, z_2 + P_3\, z_1\, z_2 + P_3\, z_1 - P_2\, z_2)\, z^2 +$$

$$+ (-z_1 - z_2 + P_1\, z_1 - P_3\, z_1\, z_2 + P_1\, z_2 + P_2\, z_1\, z_2)\, z - P_1\, z_1\, z_2\big] \frac{d\varphi}{dz} +$$

$$+ [(Q_1 + Q_2 + Q_3 + Q_4 + Q_5) z^4 + \{(-Q_1 - Q_2 - Q_3 - Q_5) z_1 +$$

$$+ (-Q_1 - Q_2 - Q_3 - Q_4) z_2 + Q_2 - Q_3\} z^3 +$$

$$+ (Q_3\, z_1\, z_2 + Q_2\, z_1\, z_2 + Q_3\, z_2 - Q_1 - Q_4 - Q_5 + Q_1\, z_1\, z_2 + Q_3\, z_1 - Q_2\, z_2 - Q_2\, z_1)\, z^2 +$$

$$+ (Q_1\, z_2 + Q_2\, z_1\, z_2 + Q_5\, z_1 + Q_1\, z_1 - Q_3\, z_1\, z_2 + Q_4\, z_2)\, z - Q_1\, z_1\, z_2]\, \varphi = 0\,.$$

Solutions for $\varphi(z)$ are constructed as power series with 6-term recurrent relations

$$k \geq 4, \qquad (Q_1 + Q_2 + Q_3 + Q_4 + Q_5)\; d_{k-4} +$$

$$+ [(k-3)(k-4) + (P_1 + P_2 + P_3 - 2)\; (k-3) +$$

$$+ (-Q_1 - Q_2 - Q_3 - Q_5)\, z_1 + (-Q_1 - Q_2 - Q_3 - Q_4)\, z_2 + Q_2 - Q_3]\; d_{k-3} +$$

$$+ [(-z_1 - z_2)\; (k-2)(k-3) + \{(1 - P_1 - P_2 - P_3)\, z_1 + (1 - P_1 - P_2 - P_3)\, z_2 + P_2 - P_3\}\; (k-2) +$$

$$+ Q_3\, z_1\, z_2 + Q_2\, z_1\, z_2 + Q_3\, z_2 - Q_1 - Q_4 - Q_5 + Q_1\, z_1\, z_2 + Q_3\, z_1 - Q_2\, z_2 - Q_2\, z_1]\; d_{k-2} +$$

$$+ [(z_1 z_2 - 1)\,(k-1)(k-2) + (2 - P_1 - P_2 z_1 + P_3 z_2 + P_2 z_1 z_2 + P_1 z_1 z_2 + P_3 z_1 z_2 + P_3 z_1 - P_2 z_2)\,(k-1) +$$

$$+ Q_1\, z_2 + Q_2\, z_1\, z_2 + Q_5\, z_1 + Q_1\, z_1 - Q_3\, z_1\, z_2 + Q_4\, z_2]\; d_{k-1} +$$

$$+ [(z_1 + z_2)\; k\,(k-1) + (-z_1 - z_2 + P_1\, z_1 - P_3\, z_1\, z_2 + P_1\, z_2 + P_2\, z_1\, z_2)\, k - Q_1\, z_1\, z_2]\; d_k +$$

$$+ [-z_1\, z_2\, (k+1)\, k - P_1\, z_1\, z_2\, (k+1)]\; d_{k+1} = 0\,. \tag{2.11}$$

Possible convergence radii are

$$R_{\text{conv}} = \left|\frac{1}{R}\right| = +1\,,\; +\infty\,,\; |z_1|\,,\; |z_2|. \tag{2.12}$$

It is readily checked that the coefficient at d_{k-4} в (2.11) vanishes identically, so in (2.11) we have 5-term recurrent relations

$$k \geq 4, \quad S_{k-3} d_{k-3} + S_{k-2} d_{k-2} + S_{k-1} d_{k-1} + S_k d_k + S_{k+1} d_{k+1} = 0\;. \tag{2.13}$$

As quantization rule we apply the known transcendency condition

$$k \geq 3, \quad S_{k-3} = 0, \qquad (k-3)(k-4) + (P_1 + P_2 + P_3 - 2)\; (k-3) +$$
$$+ (-Q_1 - Q_2 - Q_3 - Q_5)\, z_1 + (-Q_1 - Q_2 - Q_3 - Q_4)\, z_2 + Q_2 - Q_3 = 0\,, \tag{2.14}$$

which gives

$$k^2 + (2\,M + 2\,\alpha + 2\,\beta - 6)\, k - (B + 2\, i\sigma\, \nu)\, z_1 - (A + 2\, i\sigma\, \nu)\, z_2 +$$

$$+ (2\,M + 2\,\beta - 6)\,\alpha + (2\,M - 6)\,\beta - 6\,M - C - D + 2\,\nu + 9 = 0\,.$$

Whence, takin in mind the expressions for A, B, C, D:

$$A = -\frac{2\nu \left(iz_1\,\sigma + 1 + 2\,\sigma^2\right)}{z_1 - z_2}\,, \quad B = -\frac{2\nu \left(iz_2\,\sigma + 1 + 2\,\sigma^2\right)}{z_2 - z_1}\,,$$

$$C = -\,(E + ie)^2 + m^2\,, \quad D = -\,(E - ie)^2 + m^2\,, \qquad -4iEe = C - D\,,$$

we arrive at

$$k^2 + 2\,k\,(M + \alpha + \beta - 3) - 2\,i\sigma\,\nu\,(z_1 + z_2) + 2\,(M + \beta - 3)\,\alpha + 2\,(M - 3)\,\beta + \\ +9 - 6\,M - 2\,m^2 - 4\,\nu\,\sigma^2 - 2\,e^2 + 2\,E^2 = 0\,. \tag{2.15}$$

We will follow two possibilities. The first is

$$M = \sqrt{\nu^2 - e^2}\,, \quad \alpha = +\sqrt{(E + ie)^2 - m^2}\,, \quad \beta = +\sqrt{(E - ie)^2 - m^2}\,, \tag{2.16}$$

then eq. (2.15) takes the form

$$\left(\sqrt{(E + ie)^2 - m^2} + \sqrt{(E - ie)^2 - m^2} + k - 3 + \sqrt{\nu^2 - e^2}\right)^2 - \left(\nu^2 - e^2\right) = 0\,,$$

or differently

$$\left(\sqrt{(E + ie)^2 - m^2} + \sqrt{(E - ie)^2 - m^2} + k - 3\right) \times \\ \times \left(\sqrt{(E + ie)^2 - m^2} + \sqrt{(E - ie)^2 - m^2} + k - 3 + 2\sqrt{\nu^2 - e^2}\right) = 0,$$

Here there arise two equation which both are of small physical interest: (let $n = k - 3$)

$$\sqrt{(E + ie)^2 - m^2} + \sqrt{(E - ie)^2 - m^2} + n = 0, \tag{2.17}$$

$$\sqrt{(E + ie)^2 - m^2} + \sqrt{(E - ie)^2 - m^2} + n + 2\sqrt{\nu^2 - e^2} = 0. \tag{2.18}$$

Now consider t he second variant

$$M = \sqrt{\nu^2 - e^2}\,, \quad \alpha = -\sqrt{(E + ie)^2 - m^2}\,, \quad \beta = -\sqrt{(E - ie)^2 - m^2}\,, \tag{2.19}$$

then we have transcendency condition

$$\left(\sqrt{(E + ie)^2 - m^2} + \sqrt{(E - ie)^2 - m^2} - (k - 3) - \sqrt{\nu^2 - e^2}\right)^2 - \left(\nu^2 - e^2\right) = 0\,,$$

or differently

$$\times \left(\sqrt{(E + ie)^2 - m^2} + \sqrt{(E - ie)^2 - m^2} - (k - 3)\right) \times \\ \times \left(\sqrt{(E + ie)^2 - m^2} + \sqrt{(E - ie)^2 - m^2} - (k - 3) - 2\sqrt{\nu^2 - e^2}\right) = 0.$$

So we obtain two alternative equations

$$\sqrt{(E + ie)^2 - m^2} + \sqrt{(E - ie)^2 - m^2} - n = 0 \tag{2.20}$$

$$\sqrt{(E+ie)^2-m^2}+\sqrt{(E-ie)^2-m^2}-n-2\sqrt{\nu^2-e^2}=0. \qquad (2.21)$$

Interesting is only second one (2.21, it gives

$$\sqrt{(E+ie)^2-m^2}=2N-\sqrt{(E-ie)^2-m^2}, \quad N=n/2+\sqrt{\nu^2-e^2},$$

or

$$N\sqrt{E^2-m^2-e^2-2\,iEe}=-iEe+N^2\,.$$

Further we obtain $N^2(E^2-m^2-e^2-2ieE)=N^4-2ieEN^2-e^2E^2$. whence it follows the needed energy spectrum

$$E=m\sqrt{\frac{1+(e^2+N^2)/m^2}{1+e^2/N^2}}, \quad N=\frac{n}{2}+\sqrt{\nu^2-e^2}, \quad m=\frac{Mc\rho}{\hbar}\,. \qquad (2.22)$$

This spectrum coincides with that found in [10, 14], when studying the same problem for Dirac equation in Riemann space within WKB-approach.

As in previous section, we can prove that does not exist the possibility to get the above obtained energy spectrum by imposing polynomial conditions.

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