Vector Particle with Electric Quadrupole Moment in External Coulomb Field

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In the paper, the problem of vector particle with electric quadrupole moment in presence of external Coulomb field is investigated. Starting with relativistic Duffin-Kemmer theory we search solutions while diagonalizing the operators of energy, square and third projection of the total angular momentum. After separating the variables we derive the system of 10 radial equations. According to requirement of diagonalizing the spatial reflection operator, we split the system into two subsystems of 4 and 6 equations, respectively, for states with parities $P = (-1)^{j+1}$ and $P = (-1)^j$. Additional interaction terms enter both subsystems. Relativistic radial subsystem of 4 equations reduces to a second order equation which contains two irregular points $x = 0$ and $x = \infty$ of the rank 3 and 2, respectively, and four regular points. Its local Frobenius solutions near the point $x = 0$ are constructed. It is shown that there are 8-term recurrence formulas for involved power series. Condition of transcendency of solutions gives some quantization rule for energy levels which seems to be only partially physically appropriate. Relativistic radial system of 6 equations for states with the parity $P = (-1)^j$ turns out to be very complicated. In order to simplify the problem, we perform the transition to nonrelativistic approximation, so deriving two linked second order differential equations for two radial functions. We derive 4-th order equations for radial functions. There are constructed 4 different Frobenius type solutions of these equations, convergence of involved power series with 8- and 9-terms recurrence relations is studied. Transcendency condition gives the formula for energies, which does not depend on the quantum number and the parameter of quadrupole electric moment and therefore cannot describe physical spectrum correctly. Nonrelativistic analysis is performed for states with the parity $P = (-1)^j$ as well, radial equation for the main function turns out to have more simple structure of singular points. Transcendency condition leads to the formula for energies which only partially correlates with relativistic one. All the constructed solutions are exact, but they are formal because any reliable rule for quantization of energy levels is not known, the transcendency condition is solving this difficulty only partly.

We additionally apply the geometrical method based on the use of KCC-invariants. The first and the second invariants are calculated. It is shown that the different branches of the solutions converge near the singular points $\infty, -\Gamma/2$, and may either converge or diverge near the singular point $r = 0$. This correlates with the expected behavior of solutions for bound states. The explicit Lagrangians related to the geometrical problem are determined. It is shown that the Lagrangians have the arbitrariness degree up to certain terms, which may be considered as specific gauge freedom.

I. THE INTRODUCTION, INITIAL RELATIVISTIC EQUATION

It is known that in the framework of relativistic wave equation theory one can propose so-called nonminimal equations that describe particles with additional electromagnetic characteristics, with spectra of spin and mass states [1]–[5]. Within this approach, in [6]–[12] the problem of spin 1 particles with additional anomalous magnetic and quadrupole electric moments has been investigated. The equations were studied and solved for the particles in the

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external homogenous electric and magnetic fields. The equation for the vector particle in the external Coulomb field is very complicated even in the case of the ordinary particle without additional electromagnetic moments, and has not been solved in full yet. However, at nonrelativistic limit the equation for the ordinary vector particle in the Coulomb field can be solved exactly. In the present work we study the nonrelativistic problem of a vector particle with additional quadrupole moment in the external Coulomb field.

The initial equation has the following form (further we use the conventional tetrad formalism [13])

$$\left\{ i\beta^c \left[i(e^{\beta}_{(c)}\partial_\beta + \frac{1}{2} j^{ab}\gamma_{abc}(x)) \; - e' A_c \right] + \lambda \frac{e}{M} F_{\alpha\beta}(x) \bar{P} j^{\alpha\beta}(x) - M \right\} \Psi = 0\,; \tag{1}$$

related to quadrupole moment free parameter λ is dimensionless, $\bar{P}$ is a projective operator separating from the 10-component wave function the tensor component; we use notations:

$$\bar{P} = \begin{vmatrix} 0 & 0 \\ 0 & I_6 \end{vmatrix}, \qquad M = \frac{mc}{\hbar}, \qquad e' = \frac{e}{c\hbar}, \qquad \Gamma = \lambda \frac{4\alpha}{M}, \qquad \alpha = \frac{e^2}{\hbar c}\,. \tag{2}$$

In spherical tetrad [14] the equation (1) has the form

$$\left[\beta^0 (i\partial_t + \frac{\alpha}{r}) + i \left(\beta^3 \; \partial_r + \frac{1}{r}(\beta^1 j^{31} + \beta^2 j^{32}) \right) + \frac{1}{r}\Sigma_{\theta,\phi} + \frac{\Gamma}{r^2} \bar{P} j^{03} - M \right] \Phi(x) = 0\,, \tag{3}$$

where depending on angular variables operator $\Sigma_{\theta,\phi}$ is determined by the equality

$$\Sigma_{\theta,\phi} = i \; \beta^1 \partial_\theta + \beta^2 \frac{i\partial_\phi + i j^{12}\cos\theta}{\sin\theta}\,. \tag{4}$$

The components of the operator of the total angular momentum are given [14] in this basis by the formulae

$$j_1 = l_1 + \frac{\cos\phi}{\sin\theta} i j^{12}, \quad j_2 = l_2 + \frac{\sin\phi}{\sin\theta} i j^{12}, \quad j_3 = l_3\,, \quad j^{12} = \beta^1\beta^2 - \beta^2\beta^1. \tag{5}$$

Further we will use cyclic basis for Duffin-Kemmer matrices [14]:

$$\beta^0 = \begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & +i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & +i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & +i & 0 & 0 & 0 \\
0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix}, \quad
\beta^3 = \begin{vmatrix}
0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix},$$

$$\beta^1 = \frac{1}{\sqrt{2}} \begin{vmatrix}
0 & 0 & 0 & 0 & -i & 0 & +i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & +1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\
-i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix}, \quad
\beta^2 = \frac{1}{\sqrt{2}} \begin{vmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & +i & 0 & -i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +i & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & +i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i & 0 & +i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix}.$$

The matrix $ij^{12} = i(\beta^1\beta^2 - \beta^2\beta^1)$ has the diagonal structure:

$$ij^{12} = \begin{vmatrix}
0 & 0 & 0 & 0 \\
0 & t_3 & 0 & 0 \\
0 & 0 & t_3 & 0 \\
0 & 0 & 0 & t_3
\end{vmatrix}, \qquad t_3 = \begin{vmatrix}
+1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{vmatrix}.$$

Transforming 10 × 10 - matrices from Cartesian basis to cyclic one is performed by similarity transformation

$$
\Psi_{cycl} = S \Psi_{Cart}, \quad S = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & U & 0 & 0 \\ 0 & 0 & U & 0 \\ 0 & 0 & 0 & U \end{array} \right| , \quad U = \left| \begin{array}{ccc} -1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ +1/\sqrt{2} & i/\sqrt{2} & 0 \end{array} \right| ,
$$
$$
\beta^c_{cycl} = S \beta^c_{cart} S^{-1}, \quad \bar{P}_{cycl} = S \bar{P}_{cart} S^{-1} \equiv \bar{P}_{Cart} \, ; \tag{6}
$$

the form of the projective operator P does not change. Further all formulae are referred to cyclic basis.

The system of radial equations for the ordinary vector particle in Coulomb field is known [10]. To get similar system for the particle with quadrupole moment, it suffices to find the explicit form of additional term in the equation:

$$
\frac{\Gamma}{r^2} \bar{P} j^{03} = \frac{\Gamma}{r^2} \bar{P} (\beta^0 \beta^3 - \beta^3 \beta^0) \, . \tag{7}
$$

Taking into consideration identities

$$
\beta^0 \beta^3 - \beta^3 \beta^0 = \left| \begin{array}{cccccccccc}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +i \\
0 & 0 & 0 & 0 & +i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0
\end{array} \right| , \bar{P} j^{03} = \left| \begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +i \\
0 & 0 & 0 & 0 & +i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0
\end{array} \right| .
$$

II. SEPARATING THE VARIABLES IN THE RELATIVISTIC EQUATION

The most general form of 10-component wave function with quantum numbers ϵ, j, m is given below

$$
\Psi(x) = \{ \; \Psi_0(x), \; \vec{\Psi}(x), \; \vec{E}(x), \vec{H}(x) \; \} \, , \quad \Phi_0(x) = e^{-i\epsilon t} f_0(r) D_0, \quad \vec{\Phi}(x) = e^{-i\epsilon t} \left| \begin{array}{c} f_1(r) D_{-1} \\ f_2(r) D_0 \\ f_3(r) D_{+1} \end{array} \right| ,
$$
$$
\vec{E}(x) = e^{-i\epsilon t} \left| \begin{array}{c} E_1(r) D_{-1} \\ E_2(r) D_0 \\ E_3(r) D_{+1} \end{array} \right| , \quad \vec{H}(x) = e^{-i\epsilon t} \left| \begin{array}{c} H_1(r) D_{-1} \\ H_2(r) D_0 \\ H_3(r) D_{+1} \end{array} \right| , \tag{8}
$$

where we use Wigner D-functions [14]: $D_\sigma = D^j_{-m,\sigma}(\phi, \theta, 0) \, , \quad \sigma = 0, -1, +1.$

Starting with the known system of radial equations for the ordinary vector particle is [10] (let $\nu = \sqrt{j(j+1)}/\sqrt{2}$):

$$
-(\frac{d}{dr} + \frac{2}{r}) \, E_2 - \frac{\nu}{r} \, (E_1 + E_3) = m \, f_0 \, , \; +i \, (\epsilon + \frac{\alpha}{r}) \, E_1 \quad + i \, (\frac{d}{dr} + \frac{1}{r}) \, H_1 + i \frac{\nu}{r} \, H_2 = m f_1 \, ,
$$
$$
+i \, (\epsilon + \frac{\alpha}{r}) \, E_2 - i \frac{\nu}{r} \, (H_1 - H_3) = m \, f_2 \, , \; +i \, (\epsilon + \frac{\alpha}{r}) \, E_3 \quad - i \, (\frac{d}{dr} + \frac{1}{r}) \, H_3 - i \frac{\nu}{r} \, H_2 = m \, f_3 \, ;
$$
$$
-i \, (\epsilon + \frac{\alpha}{r}) \, \Phi_1 \quad + \frac{\nu}{r} \, \Phi_0 = m \, E_1 \, , \quad -i \, (\epsilon + \frac{\alpha}{r}) \, f_2 \quad - \frac{d}{dr} \, f_0 = m \, E_2 \, , \tag{9}
$$
$$
-i \, (\epsilon + \frac{\alpha}{r}) \, f_3 \quad + \frac{\nu}{r} \, f_0 = m \, E_3 \, , \quad -i \, (\frac{d}{dr} + \frac{1}{r}) \, f_1 - i \frac{\nu}{r} \, f_2 = m \, H_1 \, ,
$$
$$
+i \frac{\nu}{r} \, (f_1 - f_3) = m \, H_2 \, , \quad +i \, (\frac{d}{dr} + \frac{1}{r}) \, f_3 + i \frac{\nu}{r} f_2 = m \, H_3 \, .
$$

and accounting for the explicit form of additional term in the equation (the multiplier $e^{-i\epsilon t}$ is omitted)

$$\frac{\Gamma}{r^2}\,\bar{P}j^{03}\,\Psi=\frac{\Gamma}{r^2}\left|\begin{array}{cccccccccc}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&-i&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&+i\\0&0&0&0&+i&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&-i&0&0&0\end{array}\right|\left|\begin{array}{c}f_0D_0\\f_1D_{-1}\\f_2D_0\\f_3D_{+1}\\E_1D_{-1}\\E_2D_0\\E_3D_{+1}\\H_1D_{-1}\\H_2D_0\\H_3D_{+1}\end{array}\right|=\frac{\Gamma}{r^2}\left|\begin{array}{c}0\\0\\0\\0\\-iH_1D_{-1}\\0\\iH_3D_{+1}\\iE_1D_{-1}\\0\\-iE_3D_{+1}\end{array}\right|$$

we derive the following system

$$-(\frac{d}{dr}+\frac{2}{r})\,E_2-\frac{\nu}{r}\,(E_1+E_3)=m\,f_0\,,\;+i\,(\epsilon+\frac{\alpha}{r})\,E_1\;\;+i\,(\frac{d}{dr}+\frac{1}{r})\,H_1+i\frac{\nu}{r}\,H_2=mf_1\,,$$
$$+i\,(\epsilon+\frac{\alpha}{r})\,E_2-i\frac{\nu}{r}\,(H_1-H_3)=m\,f_2\,,\;+i\,(\epsilon+\frac{\alpha}{r})\,E_3\;\;-i\,(\frac{d}{dr}+\frac{1}{r})\,H_3-i\frac{\nu}{r}\,H_2=m\,f_3\,; \tag{10}$$

$$-i\,(\epsilon+\frac{\alpha}{r})\,f_1\;\;+\frac{\nu}{r}\,f_0-i\frac{\Gamma}{r^2}H_1=m\,E_1\,,$$
$$-i\,(\epsilon+\frac{\alpha}{r})\,f_2\;\;-\;\frac{d}{dr}\,f_0=m\,E_2\,,\;-i\,(\epsilon+\frac{\alpha}{r})\,f_3\;\;+\frac{\nu}{r}\,f_0+i\frac{\Gamma}{r^2}H_3=m\,E_3\,, \tag{11}$$

$$-i\,(\frac{d}{dr}+\frac{1}{r})\,f_1-i\frac{\nu}{r}\,f_2+i\frac{\Gamma}{r^2}E_1=m\,H_1\,,\;+i\frac{\nu}{r}\,(f_1-f_3)=m\,H_2\,,$$
$$+i\,(\frac{d}{dr}+\frac{1}{r})\,f_3+i\frac{\nu}{r}f_2-i\frac{\Gamma}{r^2}E_3=m\,H_3\,. \tag{12}$$

Additionally to operators $\vec{j}^{\,2}, j_3$ we diagonalize the operator of spatial reflection $\hat{\Pi}$. In Cartesian matrix basis β^a this operator has an ordinary form

$$\hat{\Pi}=\left|\begin{array}{cccc}1&0&0&0\\0&-I&0&0\\0&0&-I&0\\0&0&0&+I\end{array}\right|\hat{P}\,,\quad\hat{P}\Psi(\vec{r})=\Psi(-\vec{r})\,. \tag{13}$$

After transition it to spherical tetrad and cyclic representation of the matrix β^a we obtain

$$\hat{\Pi}'=\left|\begin{array}{cccc}1&0&0&0\\0&\Pi_3&0&0\\0&0&\Pi_3&0\\0&0&0&-\Pi_3\end{array}\right|\hat{P}\,,\quad\Pi_3=\left|\begin{array}{ccc}0&0&-1\\0&-1&0\\-1&0&0\end{array}\right|. \tag{14}$$

The eigenvalues equation $\hat{\Pi}'\Psi=P\Psi$ (accounting for the known property $\hat{P}D_\sigma=(-1)^jD_{-\sigma}$) yields the set of algebraic equations:

$$(-1)^jf_0=Pf_0\,,\;(-1)^jf_3=Pf_1\,,\;(-1)^jf_2=Pf_2\,,\;(-1)^jf_1=Pf_3\,,$$
$$(-1)^jE_3=PE_1\,,\;(-1)^jE_2=PE_2\,,\;(-1)^jE_1=PE_3\,,$$
$$(-1)^jH_3=-PH_1\,,\;(-1)^jH_2=-PH_2\,,\;(-1)^jH_1=-PH_3\,. \tag{15}$$

This system has two solutions:

$$P=(-1)^{j+1},\qquad f_0=0\,,\;f_3=-f_1\,,\;f_2=0\,,\;\;E_3=-E_1\,,\;E_2=0\,,\;H_3=H_1\,; \tag{16}$$

$$P=(-1)^j,\qquad f_3=+f_1\,,\;\;E_3=+E_1\,,\;H_3=-H_1\,,\;H_2=0\,. \tag{17}$$

It easy to check that these restrictions are compatible with the above radial equations.

For the parity $P=(-1)^{j+1}$ we get

$$f_0=0\,,\; f_3=-f_1\,,\; f_2=0\,,\; E_3=-E_1\,,\; E_2=0\,,\; H_3=H_1\,,$$
$$0=0\,,\quad +i\,(\epsilon+\frac{\alpha}{r})\;E_1\;\;+i\,(\frac{d}{dr}+\frac{1}{r})\;H_1+i\frac{\nu}{r}\;H_2=mf_1\,,$$
$$0=0\,,\quad -i\,(\epsilon+\frac{\alpha}{r})\;E_1\;\;-i\,(\frac{d}{dr}+\frac{1}{r})\;H_3-i\frac{\nu}{r}\;H_2=-m\;f_1\;;$$
$$-i\,(\epsilon+\frac{\alpha}{r})\;f_1\;\;-i\frac{\Gamma}{r^2}H_1=m\;E_1\;,\quad 0=0\,,\quad i\,(\epsilon+\frac{\alpha}{r})\;f_1\;\;+i\frac{\Gamma}{r^2}H_1=-m\;E_1\;,$$
$$-i\,(\frac{d}{dr}+\frac{1}{r})\;f_1+i\frac{\Gamma}{r^2}E_1=m\;H_1\;,\;+2\,i\frac{\nu}{r}\;f_1=m\;H_2\;,\quad -i\,(\frac{d}{dr}+\frac{1}{r})\;f_1+i\frac{\Gamma}{r^2}E_1=m\;H_1;$$

so we have only four equations:

$$+i\left(\epsilon+\frac{\alpha}{r}\right)E_1+i\left(\frac{d}{dr}+\frac{1}{r}\right)H_1+i\frac{\nu}{r}H_2=mf_1\,,$$
$$-i\left(\epsilon+\frac{\alpha}{r}\right)\;f_1\;\;-i\frac{\Gamma}{r^2}H_1=m\;E_1\,,\tag{18}$$
$$-i\left(\frac{d}{dr}+\frac{1}{r}\right)\;f_1+i\frac{\Gamma}{r^2}E_1=m\;H_1\,,\quad 2i\frac{\nu}{r}f_1=mH_2\,.$$

We exclude variables H_1,H_2 from the first and second equations:

$$+i\left(\epsilon+\frac{\alpha}{r}\right)mE_1+i\left(\frac{d}{dr}+\frac{1}{r}\right)\left[-i\left(\frac{d}{dr}+\frac{1}{r}\right)\;f_1+i\frac{\Gamma}{r^2}E_1\right]-\frac{2\nu^2}{r^2}f_1=m^2f_1\,,$$
$$-i\left(\epsilon+\frac{\alpha}{r}\right)\;mf_1\;\;-i\frac{\Gamma}{r^2}\left[-i\left(\frac{d}{dr}+\frac{1}{r}\right)\;f_1+i\frac{\Gamma}{r^2}E_1\right]=m^2E_1\,.\tag{19}$$

The Second equation permits to express E_1 through f_1:

$$-i\left(\epsilon+\frac{\alpha}{r}\right)\;mf_1\;\;-\frac{\Gamma}{r^2}\left(\frac{d}{dr}+\frac{1}{r}\right)\;f_1=(m^2-\frac{\Gamma^2}{r^4})E_1\,;\tag{20}$$

consequently the function E_1 can be excluded. So, we arrive at a 2-nd order equation for the main function f_1:

$$\frac{d^2f_1}{dr^2}+\left[-\frac{2\,r\,m}{mr^2+\Gamma}-\frac{2\,r\,m}{mr^2-\Gamma}+\frac{6}{r}\right]\frac{df_1}{dr}+$$
$$+\left[\frac{2\,\epsilon\,\alpha}{r}+\frac{-2\,\nu^2+\alpha^2+4}{r^2}+\frac{2\,i\Gamma\epsilon}{mr^3}+\frac{\Gamma\,(\Gamma m+i\alpha)}{mr^4}+\frac{2\,\nu^2\Gamma^2}{m^2r^6}-\right.$$
$$\left.-m^2+\epsilon^2+\frac{2\,m\,(i\epsilon\,r+i\alpha-1)}{mr^2+\Gamma}-\frac{2\,m\,(i\epsilon\,r+i\alpha+1)}{mr^2-\Gamma}\right]f_1=0\,.\tag{21}$$

It has irregular singular point $r=0$ of the rank 3, irregular point $r=\infty$ of the rank 2, and four regular points which are determined by the roots of the equation, $(r^2-\Gamma/m)(r^2+\Gamma/m)=0\,.$

Let us consider states with other parity:

$$P=(-1)^j,\quad f_3=+f_1\;,\quad E_3=+E_1\;,\;H_3=-H_1\;,\;H_2=0\,;\tag{22}$$

at such restrictions the radial system takes the form

$$-(\frac{d}{dr}+\frac{2}{r})\;E_2-2\,\frac{\nu}{r}\;E_1=m\;f_0\,,\quad +i\,(\epsilon+\frac{\alpha}{r})\;E_1\;\;+i\,(\frac{d}{dr}+\frac{1}{r})\;H_1=mf_1\,,$$
$$+i\,(\epsilon+\frac{\alpha}{r})\;E_2-2\,i\frac{\nu}{r}\;H_1=m\;f_2\,,\quad +i\,(\epsilon+\frac{\alpha}{r})\;E_1\;\;+i\,(\frac{d}{dr}+\frac{1}{r})\;H_1=m\;f_1\,;$$
$$-i\,(\epsilon+\frac{\alpha}{r})\;f_1\;\;+\frac{\nu}{r}\;f_0-i\frac{\Gamma}{r^2}H_1=m\;E_1\,,\quad -i\,(\epsilon+\frac{\alpha}{r})\;f_2\;\;-\;\frac{d}{dr}\;f_0=m\;E_2\,,$$
$$-i\,(\epsilon+\frac{\alpha}{r})\;f_1\;\;+\frac{\nu}{r}\;f_0-i\frac{\Gamma}{r^2}H_1=m\;E_1\,,\quad -i\,(\frac{d}{dr}+\frac{1}{r})\;f_1-i\frac{\nu}{r}\;f_2+i\frac{\Gamma}{r^2}E_1=m\;H_1\,,$$
$$0=0\,,\quad +i\,(\frac{d}{dr}+\frac{1}{r})\;f_1+i\frac{\nu}{r}f_2-i\frac{\Gamma}{r^2}E_1=-m\;H_1\,;$$

so we arrive at six different equations:

$P = (-1)^j$,

$$-\left(\frac{d}{dr}+\frac{2}{r}\right)E_2 - 2\,\frac{\nu}{r}\,E_1 = m\,f_0\,, \quad +i\,\left(\epsilon+\frac{\alpha}{r}\right)E_1 + i\left(\frac{d}{dr}+\frac{1}{r}\right)H_1 = m f_1\,,$$
$$+i\left(\epsilon+\frac{\alpha}{r}\right)E_2 - 2\,i\frac{\nu}{r}\,H_1 = m\,f_2\,; \quad -i\left(\epsilon+\frac{\alpha}{r}\right)f_1 + \frac{\nu}{r}\,f_0 - i\frac{\Gamma}{r^2}H_1 = m\,E_1\,, \tag{23}$$
$$-i\,\left(\epsilon+\frac{\alpha}{r}\right)f_2 \;-\; \frac{d}{dr}\,f_0 = m\,E_2\,, \quad +i\left(\frac{d}{dr}+\frac{1}{r}\right)f_1 + i\frac{\nu}{r}f_2 - i\frac{\Gamma}{r^2}E_1 = -m\,H_1\,.$$

III. STATES WITH PARITY $P = (-1)^{j+1}$

In eqs (21) we introduce new (dimensionless) variables

$$x = mr, \quad E = \frac{\epsilon}{m}, \quad \gamma = m\Gamma\,, \tag{24}$$

so obtaining a simpler representation

$$\frac{d^2 f_1}{dx^2} + \left[-\frac{2\,x}{x^2+\gamma} - \frac{2x}{x^2-\gamma} + \frac{6}{x}\right]\frac{df_1}{dx} +$$
$$+\left[(E^2-1) + \frac{2E\alpha}{x} + \frac{-2\nu^2+\alpha^2+4}{x^2} + \frac{2i\gamma E}{x^3} + \frac{\gamma(\gamma+i\alpha)}{x^4} + \frac{2\nu^2\gamma^2}{x^6} + \right.$$
$$\left. + \frac{2(iEx+i\alpha-1)}{x^2+\gamma} - \frac{2(iEx+i\alpha+1)}{x^2-\gamma}\right] f_1 = 0\,. \tag{25}$$

Here we have the singular points

$$x = 0\,, \quad \text{Rank} = 3\,, \qquad x = \infty\,, \quad \text{Rank} = 2\,,$$
$$x = -\sqrt{\gamma}\,,\ +\sqrt{\gamma}\,,\ -i\sqrt{\gamma}\,,\ +i\sqrt{\gamma}, \quad \text{Rank} = 1\,. \tag{26}$$

The equation is suitable to be represented in the symbolic form:

$$\frac{d^2 f_1}{dx^2} + \left[-\frac{2\,x}{x^2+\gamma} - \frac{2x}{x^2-\gamma} + \frac{6}{x}\right]\frac{df_1}{dx} +$$
$$+\left[E^2 - 1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \frac{a_4}{x^4} + \frac{a_6}{x^6} + \frac{L}{x^2+\gamma} + \frac{N}{x^2-\gamma}\right] f_1 = 0\,. \tag{27}$$

We construct its Frobenius type solutions near the point $x = 0$ in the form

$$f_1(x) = e^{Dx} x^A e^{B/x} e^{C/x^2} F(x)\,,$$

$$\frac{d^2 F}{dx^2} + \left[-\frac{2\,x}{x^2+\gamma} - \frac{2\,x}{x^2-\gamma} + \frac{2\,A+6}{x} - \frac{2\,B}{x^2} - \frac{4\,C}{x^3} + 2\,D\right]\frac{dF}{dx} +$$
$$+\left[(E^2+D^2-1) + \frac{2\,AD + 6\,D + a_1}{x} + \frac{A^2 + 5\,A - 2\,BD + a_2}{x^2} + \frac{-2\,AB - 4\,B - 4\,CD + a_3}{x^3} + \right.$$
$$+ \frac{-4\,AC + B^2 - 6\,C + a_4}{x^4} + \frac{4\,BC}{x^5} + \frac{4\,C^2 + a_6}{x^6} +$$
$$\left. + \frac{-2\,A\gamma - 2\,Bx - 4\,C - 2\,D\gamma x + L\gamma}{\gamma\,(x^2+\gamma)} + \frac{-2\,A\gamma + 2\,Bx + 4\,C - 2\,D\gamma x + N\gamma}{\gamma\,(x^2-\gamma)}\right] F = 0\,.$$

Let us impose the restrictions:

$$E^2 + D^2 - 1 = 0 \quad \Rightarrow \quad D = -\sqrt{1-E^2}\,, +\sqrt{1-E^2}$$
$$4\,C^2 + a_6 = 0 \quad \Rightarrow \quad C = \frac{\delta}{2}\,\sqrt{-a_6} = \delta\frac{1}{2}\,\sqrt{2\nu^2(-\gamma^2)}\,, \quad \delta = \pm 1\,,$$
$$4\,BC = 0 \quad \Longrightarrow \quad B = 0\,, \tag{28}$$
$$-4\,AC + B^2 - 6\,C + a_4 \quad \Rightarrow \quad A = -\frac{3}{2} + \frac{1}{4}\,\frac{a_4}{C} = -\frac{3}{2} + \frac{\delta}{2}\,\frac{\gamma^2 + i\gamma\alpha}{\sqrt{2\nu^2(-\gamma^2)}}\,.$$

By physical ground the parameter γ has to be imaginary, so we should make a substitution $i\gamma \Longrightarrow \gamma$, to have regular behavior at $x=0$ the parameter C should be taken as negative one. To describe bound states, we will use following expression for the parameters:

$$A=-\frac{3}{2}-\frac{1}{2}\,\frac{\gamma^2+i\gamma\alpha}{\sqrt{2\nu^2(-\gamma^2)}}\,,\quad B=0\,,\quad C=-\frac{1}{2}\,\sqrt{2\nu^2(-\gamma^2)}<0\,,\quad D=-\sqrt{1-E^2}\,. \tag{29}$$

The equation is simplified:

$$\begin{gathered}\frac{d^2F}{dx^2}+\left[-\frac{2\,x}{x^2+\gamma}-\frac{2\,x}{x^2-\gamma}+\frac{2\,A+6}{x}-\frac{2\,B}{x^2}-\frac{4\,C}{x^3}+2\,D\right]\frac{dF}{dx}+\\ +\left[\frac{2\,AD+6\,D+a_1}{x}+\frac{A^2+5\,A-2\,BD+a_2}{x^2}+\frac{-2\,AB-4\,B-4\,CD+a_3}{x^3}+\right.\\ \left.+\frac{-2\,A\gamma-2\,Bx-4\,C-2\,D\gamma x+L\gamma}{\gamma\,(x^2+\gamma)}+\frac{-2\,A\gamma+2\,Bx+4\,C-2\,D\gamma x+N\gamma}{\gamma\,(x^2-\gamma)}\right]F=0\,.\end{gathered} \tag{30}$$

Multiplying this equation by $x^3(x^2-\gamma)(x^2+\gamma)$, we get

$$\begin{gathered}\left[x^7-x^3\gamma^2\right]\frac{d^2F}{dx^2}+\\ +\left[2D\,x^7+(2+2\,A)\,x^6-2B\,x^5-4C\,x^4-2D\gamma^2\,x^3-2\,\gamma^2\,(3+A)\,x^2+2B\gamma^2\,x+4\,C\gamma^2\right]\frac{dF}{dx}+\\ +\left[(2\,AD+2\,D+a_1)\,x^6+\left(A^2+A-2\,BD+L+N+a_2\right)x^5+\right.\\ +(-2\,AB-4\,CD+a_3)\,x^4+(8\,C-L\gamma+N\gamma)\,x^3+\gamma^2\,(-2\,AD-6\,D-a_1)\,x^2+\\ \left.+\gamma^2\left(-A^2-5\,A+2\,BD-a_2\right)x+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\right]F=0\,.\end{gathered}$$

Solutions for $F(x)$ we construct as power series:

$$F=\sum_{n=0}^{\infty}d_nx^n,\quad \frac{dF}{dx}=\sum_{n=1}^{\infty}nd_nx^{n-1},\quad \frac{d^2F}{dx^2}=\sum_{n=2}^{\infty}n(n-1)d_nx^{n-2};$$

we get

$$\begin{gathered}\sum_{k=7}^{\infty}(k-5)(k-6)d_{k-5}x^k-\gamma^2\sum_{k=3}^{\infty}(k-1)(k-2)d_{k-1}x^k+\\ +2D\sum_{k=7}^{\infty}(k-6)d_{k-6}x^k+(2+2\,A)\sum_{k=6}^{\infty}(k-5)d_{k-5}x^k-2B\sum_{k=5}^{\infty}(k-4)d_{k-4}x^k-4C\sum_{k=4}^{\infty}(k-3)d_{k-3}x^k-\\ -2D\gamma^2\sum_{k=3}^{\infty}(k-2)d_{k-2}x^k-2\,\gamma^2\,(3+A)\sum_{k=2}^{\infty}(k-1)d_{k-1}x^k+2B\gamma^2\sum_{k=1}^{\infty}kd_kx^k+4\,C\gamma^2\sum_{k=0}^{\infty}(k+1)d_{k+1}x^k+\\ +(2\,AD+2\,D+a_1)\sum_{k=6}^{\infty}d_{k-6}x^k+\left(A^2+A-2\,BD+L+N+a_2\right)\sum_{k=5}^{\infty}d_{k-5}x^k+\\ +(-2AB-4CD+a_3)\sum_{k=4}^{\infty}d_{k-4}x^k+(8C-L\gamma+N\gamma)\sum_{k=3}^{\infty}d_{k-3}x^k+\gamma^2\,(-2AD-6D-a_1)\sum_{k=2}^{\infty}d_{k-2}x^k+\\ +\gamma^2\left(-A^2-5\,A+2\,BD-a_2\right)\sum_{k=1}^{\infty}d_{k-1}x^k+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\sum_{k=0}^{\infty}d_kx^k=0\,.\end{gathered}$$

Equating coefficients at the same powers of x to zero, we derive recurrence formulae:

$$k=0\,,\quad 4\,C\,d_1+(2\,AB+4\,B+4\,CD-a_3)\;d_0=0\,,$$
$$k=1\,,\quad 2B\,d_1+8\,C\,d_2+\left(-A^2-5\,A+2\,BD-a_2\right)\;d_0+(2\,AB+4\,B+4\,CD-a_3)\;d_1=0\,,$$
$$k=2\,,\quad -2\;(3+A)\;d_1+2B\,2d_2+12\,C\,d_3+$$
$$+\,(-2\,AD-6\,D-a_1)\;d_0+\left(-A^2-5\,A+2\,BD-a_2\right)\;d_1+(2\,AB+4\,B+4\,CD-a_3)\;d_2=0\,,$$
$$k=3\quad -2\,\gamma^2\,d_2-2D\gamma^2\,d_1-4\,\gamma^2\,(3+A)\;d_2+6B\gamma^2\,d_3+16\,C\gamma^2\,d_4+$$
$$+\,(8\,C-L\gamma+N\gamma)\;d_0+\gamma^2\,(-2\,AD-6\,D-a_1)\;d_1+$$
$$+\gamma^2\left(-A^2-5\,A+2\,BD-a_2\right)\;d_2+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\;d_3=0\,,$$
$$k=4\,,\quad -6\gamma^2\,d_3-4C\,d_1-4D\gamma^2\,d_2-6\,\gamma^2\,(3+A)\;d_3+8B\gamma^2\,d_4+20\,C\gamma^2\,d_5+$$
$$+\,(-2\,AB-4\,CD+a_3)\;d_0+(8\,C-L\gamma+N\gamma)\;d_1+\gamma^2\,(-2\,AD-6\,D-a_1)\;d_2+$$
$$+\gamma^2\left(-A^2-5\,A+2\,BD-a_2\right)\;d_3+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\;d_4=0\,,$$
$$k=5\,,\quad -12\gamma^2\,d_4-2B\,d_1-8C\,d_2-6D\gamma^2\,d_3-8\,\gamma^2\,(3+A)\;d_4+10B\gamma^2\,d_5+24\,C\gamma^2\,d_6+$$
$$+\left(A^2+A-2\,BD+L+N+a_2\right)\;d_0+(-2\,AB-4\,CD+a_3)\;d_1+(8\,C-L\gamma+N\gamma)\;d_2+$$
$$+\gamma^2\,(-2\,AD-6\,D-a_1)\;d_3+\gamma^2\left(-A^2-5\,A+2\,BD-a_2\right)\;d_4+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\;d_5=0\,,$$
$$k=6\,,\quad -20\gamma^2\,d_5+(2+2\,A)\;d_1-4B\,d_2-12C\,d_3-8D\gamma^2\,d_4-10\,\gamma^2\,(3+A)\;d_5+12B\gamma^2\,d_6+28\,C\gamma^2\,d_7+$$
$$+\,(2\,AD+2\,D+a_1)\;d_0+\left(A^2+A-2\,BD+L+N+a_2\right)\;d_1+$$
$$+\,(-2\,AB-4\,CD+a_3)\;d_2+(8\,C-L\gamma+N\gamma)\;d_3+\gamma^2\,(-2\,AD-6\,D-a_1)\;d_4+$$
$$+\gamma^2\left(-A^2-5\,A+2\,BD-a_2\right)\;d_5+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\;d_6=0\,,$$
$$k=7\,,\quad 2d_2-30\gamma^2\,d_6+2D\,d_1+2\;(2+2\,A)\;d_2-6B\,d_3-16C\,d_4-$$
$$-10D\gamma^2\,d_5-12\,\gamma^2\,(3+A)\;d_6+14B\gamma^2\,d_7+32\,C\gamma^2\,d_8+$$
$$(2\,AD+2\,D+a_1)\;d_1+\left(A^2+A-2\,BD+L+N+a_2\right)\;d_2+$$
$$+\,(-2\,AB-4\,CD+a_3)\;d_3+(8\,C-L\gamma+N\gamma)\;d_4+\gamma^2\,(-2\,AD-6\,D-a_1)\;d_5+$$
$$+\gamma^2\left(-A^2-5\,A+2\,BD-a_2\right)\;d_6+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\;d_7=0\,.$$

Thus, we find 8-term recurrence relations:

$$\underline{k=6,7,8,9,...}\qquad [2D\,(k-6)+(2\,AD+2\,D+a_1)]\;d_{k-6}+$$
$$+\left[(k-5)(k-6)+(2+2\,A)\;(k-5)+\left(A^2+A-2\,BD+L+N+a_2\right)\right]\;d_{k-5}+$$
$$+\,[-2B\,(k-4)+(-2\,AB-4\,CD+a_3)]\;d_{k-4}+$$
$$+\,[-4C\,(k-3)+(8\,C-L\gamma+N\gamma)]\;d_{k-3}+\left[-2D\gamma^2\,(k-2)+\gamma^2\,(-2\,AD-6\,D-a_1)\right]\;d_{k-2}+$$
$$+\left[-\gamma^2\,(k-1)(k-2)-2\,\gamma^2\,(3+A)\;(k-1)+\gamma^2\left(-A^2-5\,A+2\,BD-a_2\right)\right]\;d_{k-1}+$$
$$+\left[2B\gamma^2\,k+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\right]\;d_k+4\,C\gamma^2\,(k+1)d_{k+1}=0\,.$$

Shortly, these relations can be written as

$$P_{k-6}c_{k-6}+P_{k-5}c_{k-5}+P_{k-4}c_{k-4}+P_{k-3}c_{k-3}+$$
$$+P_{k-2}c_{k-2}+P_{k-1}c_{k-1}+P_kc_k+P_{k+1}c_{k+1}=0\,. \tag{31}$$

Applying the Poincaré-Perrone method, let us analyze the convergence radius of the power series. To do this, we divide the last relation by $d_{k-6}k^2$

$$[2D\,(k-6)+(2\,AD+2\,D+a_1)]\;+$$
$$+\left[(k-5)(k-6)+(2+2\,A)\;(k-5)+\left(A^2+A-2\,BD+L+N+a_2\right)\right]\;\frac{d_{k-5}}{d_{k-6}}+$$

$$+\left[-2B\,(k-4)+(-2\,AB-4\,CD+a_3)\right]\,\frac{d_{k-4}}{d_{k-5}}\,\frac{d_{k-5}}{d_{k-6}}+$$
$$+\left[-4C\,(k-3)+(8\,C-L\gamma+N\gamma)\right]\,\frac{d_{k-3}}{d_{k-4}}\,\frac{d_{k-4}}{d_{k-5}}\,\frac{d_{k-5}}{d_{k-6}}+$$
$$+\left[-2D\gamma^2\,(k-2)+\gamma^2\,(-2\,AD-6\,D-a_1)\right]\,\frac{d_{k-2}}{d_{k-3}}\,\frac{d_{k-3}}{d_{k-4}}\,\frac{d_{k-4}}{d_{k-5}}\,\frac{d_{k-5}}{d_{k-6}}+$$
$$+\left[-\gamma^2(k-1)(k-2)-2\gamma^2\,(3+A)\,(k-1)+\gamma^2\left(-A^2-5A+2BD-a_2\right)\right]\times$$
$$\times\frac{d_{k-1}}{d_{k-2}}\frac{d_{k-2}}{d_{k-3}}\frac{d_{k-3}}{d_{k-4}}\frac{d_{k-4}}{d_{k-5}}\frac{d_{k-5}}{d_{k-6}}+$$
$$+\left[2B\gamma^2\,k+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\right]\,\frac{d_k}{d_{k-1}}\,\frac{d_{k-1}}{d_{k-2}}\,\frac{d_{k-2}}{d_{k-3}}\,\frac{d_{k-3}}{d_{k-4}}\,\frac{d_{k-4}}{d_{k-5}}\,\frac{d_{k-5}}{d_{k-6}}+$$
$$+4\,C\gamma^2\,(k+1)\,\frac{d_{k+1}}{d_k}\,\frac{d_k}{d_{k-1}}\,\frac{d_{k-1}}{d_{k-2}}\,\frac{d_{k-2}}{d_{k-3}}\,\frac{d_{k-3}}{d_{k-4}}\,\frac{d_{k-4}}{d_{k-5}}\,\frac{d_{k-5}}{d_{k-6}}=0\,,$$

and tend $k\to\infty$.As a result we get the algebraic equation for R:

$$R-\gamma^2R^5=0\qquad\Rightarrow\qquad R=0\,,\ \pm\frac{1}{\sqrt{\gamma}}\,,\ \pm\frac{1}{\sqrt{-\gamma}}\,.$$

Modulus of the parameter R determines the possible convergence radii.

$$R=\lim_{k\to\infty}\frac{d_{k-5}}{d_{k-6}},\qquad R_{\text{conv}}=\left|\frac{1}{R}\right|=|\sqrt{\gamma}|\,,\ +\infty\,. \tag{32}$$

As a quantization rule we use the restrictions separating transcendent Frobenius functions (see (31))

$$P_{k-6}=0\ \Longrightarrow\ 2D\,(k-6)+2\,AD+2\,D+a_1=0\,,\quad k\geq 6\,, \tag{33}$$

where

$$A=-\frac{3}{2}-\frac{1}{2}\,\frac{\gamma^2+i\gamma\alpha}{\sqrt{2\nu^2(-\gamma^2)}}\,,\quad D=-\sqrt{1-E^2}\,,\quad a_1=2E\alpha\,.$$

Taking into account that the parameter γ is imaginary one, we make the change $i\gamma\Longrightarrow\gamma$, then

$$A=-\frac{3}{2}-\frac{1}{2}\,\frac{-\gamma^2+\gamma\alpha}{\sqrt{2\nu^2\gamma^2}}\,,\quad D=-\sqrt{1-E^2}\,,\quad a_1=2E\alpha\,, \tag{34}$$

and the transcendency condition takes the form (let $k-6=n,\ n=0,1,2,...$)

$$-\sqrt{1-E^2}\,n+\left(\frac{3}{2}+\frac{\gamma\alpha-\gamma^2}{2\sqrt{l(l+1)\gamma^2}}\right)\sqrt{1-E^2}-\sqrt{1-E^2}+E\alpha=0\,,$$

or

$$\alpha E=\sqrt{1-E^2}\left(n-1/2-\frac{\gamma\alpha-\gamma^2}{2\sqrt{\gamma^2l(l+1)}}\right). \tag{35}$$

Depending on the parameter γ sign, there arise two different equations:

$$\gamma>0,\quad \alpha E=\sqrt{1-E^2}\left(n-1/2-\frac{\alpha-\gamma}{2\sqrt{l(l+1)}}\right), \tag{36}$$

$$\gamma<0,\quad \alpha E=\sqrt{1-E^2}\left(n-1/2+\frac{\alpha-\gamma}{2\sqrt{l(l+1)}}\right). \tag{37}$$

The structure of these equations is the same, so the energy spectra are similar but different:

$$\begin{gathered}\alpha E=\sqrt{1-E^2}N\quad\Longrightarrow\quad E=\frac{1}{\sqrt{1+\frac{\alpha^2}{N^2}}}\,;\\ \gamma>0\,,\quad N=n-1/2-\frac{\alpha-\gamma}{2\sqrt{l(l+1)}}\,;\quad \gamma<0\,,\quad N=n-1/2+\frac{\alpha-\gamma}{2\sqrt{l(l+1)}}\,.\end{gathered} \tag{38}$$

IV. THE CASE OF MINIMAL $j = 0$

Let us consider the case of minimal value of the total momentum $j = 0$. The relevant substitution for the corresponding wave function should be used

$$\Phi_0(x) = e^{-i\epsilon t}\, f_0(r)\,, \quad \vec{\Phi}(x) = e^{-i\epsilon t} \begin{vmatrix} 0 \\ f_2(r) \\ 0 \end{vmatrix}, \quad \vec{E}(x) = e^{-i\epsilon t} \begin{vmatrix} 0 \\ E_2(r) \\ 0 \end{vmatrix}, \quad \vec{H}(x) = e^{-i\epsilon t} \begin{vmatrix} 0 \\ H_2(r) \\ 0 \end{vmatrix}. \tag{39}$$

Corresponding equations follow from general at the restrictions

$$\nu = 0\,, \quad f_1 = f_3 = 0\,, \quad E_1 = E_3 = 0\,, \quad H_1 = H_3 = 0\,. \tag{40}$$

So, we get

$$-(\frac{d}{dr} + \frac{2}{r})E_2 = mf_0\,, \quad +i(\epsilon + \frac{\alpha}{r})E_2 = mf_2\,, \quad 0 = 0\,, \quad 0 = 0\,,$$
$$-i(\epsilon + \frac{\alpha}{r})f_2 - \frac{d}{dr}f_0 = mE_2\,, \quad 0 = 0\,, \quad 0 = 0\,, \quad 0 = mH_2\,, \quad 0 = 0\,.$$

In the case of minimal $j = 0$, the electric quadrupole moment does not manifest itself:

$$\begin{gathered} -\left(\frac{d}{dr} + \frac{2}{r}\right) E_2 = mf_0\,, \quad +i\left(\epsilon + \frac{\alpha}{r}\right) E_2 = mf_2\,, \\ -i\left(\epsilon + \frac{\alpha}{r}\right) f_2 - \frac{d}{dr}f_0 = mE_2\,, \quad H_2 = 0\,. \end{gathered} \tag{41}$$

After excluding the variables f_2, E_2 we obtain a 2-nd order equation for function $f_0(r)$:

$$\frac{d^2 f_0}{dr^2} + \left(\frac{4}{r} + \frac{-m+\epsilon}{mr - \epsilon r - \alpha} + \frac{-m-\epsilon}{mr + \epsilon r + \alpha}\right)\frac{df_0}{dr} + \left(\frac{2\,\epsilon\,\alpha}{r} + \frac{\alpha^2}{r^2} - m^2 + \epsilon^2\right) f_0 = 0\,,$$

or in the dimensionless variables $x = mr, E = \epsilon/m$:

$$\frac{d^2 f_0}{dx^2} + \left(\frac{4}{x} + \frac{-1+E}{x - Ex - \alpha} + \frac{-1-E}{x + Ex + \alpha}\right)\frac{df_0}{dx} + \left(\frac{2\,E\alpha}{x} + \frac{\alpha^2}{x^2} + E^2 - 1\right) f_0 = 0\,. \tag{42}$$

The equation has three regular singular points and one irregular point of the rank 2 at infinity.

Alternatively, excluding the variables f_0, f_2 we get the simpler equation for the main function E_2:

$$\frac{d^2 E_2}{dr^2} + \frac{2}{r}\frac{dE_2}{dr} + \left(\epsilon^2 - m^2 + \frac{2\,\epsilon\,\alpha}{r} + \frac{\alpha^2 - 2}{r^2}\right) E_2 = 0\,.$$

The last equation being transformed to the variable $y = 2\sqrt{m^2 - \epsilon^2}r$ takes the form

$$y\frac{d^2 E_2}{dy^2} + 2\,\frac{dE_2}{dy} + \left(-\frac{1}{4}\,y + \frac{\alpha^2 - 2}{y} + \frac{\epsilon\,\alpha}{\sqrt{m^2 - \epsilon^2}}\right) .E_2 = 0\,.$$

Its solutions are constructed in confluent hypergeometric functions according to standard procedure:

$$E_2 = y^A e^{By} F(y)\,, \quad y\frac{d^2 F}{dy^2} + (2\,A + 2 + 2\,By)\,\frac{dF}{dy} +$$

$$+\left[\left(B^2 - \frac{1}{4}\right) y + \frac{A^2 + A + \alpha^2 - 2}{y} + 2\,AB + 2\,B + \frac{\epsilon\,\alpha}{\sqrt{m^2 - \epsilon^2 +}}\right] F = 0\,.$$

Imposing on the parameters evident restrictions

$$A = -\frac{1}{2} \pm \frac{1}{2}\sqrt{9 - 4\alpha^2}\,, \quad B = -\frac{1}{2}\,,$$

we reduce the task to simplify the problem to confluent hypergeometric equation

$$y\frac{d^2F}{dy^2} + (2\,A + 2 - y)\,\frac{dF}{dy} + \left(-1 - A + \frac{\epsilon\,\alpha}{\sqrt{m^2 - \epsilon^2}}\right), = 0$$

with parameters

$$a = 1 + A - \frac{\epsilon\,\alpha}{\sqrt{m^2 - \epsilon^2}}\,, \qquad c = 2A + 2\,.$$

The quantization condition is chosen as usually:

$$a = \frac{1}{2}(1 + \sqrt{9 - 4\alpha^2}) - \frac{\epsilon\alpha}{\sqrt{m^2 - \epsilon^2}} = -n$$

so deriving the formula for energy levels

$$\epsilon = \frac{m}{\sqrt{1 + \alpha^2/N^2}}, \qquad N = \frac{1}{2}(1 + \sqrt{9 - 4\alpha^2}) + n,\ n = 0, 1, 2, ... \tag{43}$$

V. NONRELATIVISTIC APPROXIMATION, $P = (-1)^{j+1}, j = 1, 2, 3, ...$

Let us perform nonrelativistic approximation in radial the system (44):

$$\begin{gathered} P = (-1)^{j+1},, \quad i\left(\epsilon + \frac{\alpha}{r}\right)E_1 + i\left(\frac{d}{dr} + \frac{1}{r}\right)H_1 + i\frac{\nu}{r}H_2 = mf_1\,, \\ -i\left(\epsilon + \frac{\alpha}{r}\right)\,f_1\ - i\frac{\Gamma}{r^2}H_1 = m\,E_1\,, \\ -i\left(\frac{d}{dr} + \frac{1}{r}\right)\,f_1 + i\frac{\Gamma}{r^2}E_1 = m\,H_1\,, \quad 2i\frac{\nu}{r}f_1 = mH_2\,. \end{gathered} \tag{44}$$

By using the third and forth equations we exclude nondynamical variables H_1, H_2:

$$\begin{gathered} i(\epsilon + \frac{\alpha}{r})E_1 + \frac{1}{m}(\frac{d}{dr} + \frac{1}{r})\left[(\frac{d}{dr} + \frac{1}{r})f_1 - \frac{\Gamma}{r^2}E_1\right] - \frac{2\nu^2}{mr^2}f_1 = mf_1\,, \\ -i(\epsilon + \frac{\alpha}{r})f_1 - \frac{\Gamma}{mr^2}\left[(\frac{d}{dr} + \frac{1}{r})f_1 - \frac{\Gamma}{r^2}E_1\right] = mE_1\,. \end{gathered}$$

Big and small components are introduced by relations

$$f_1 = (\Psi_1 + \psi_1), \quad iE_1 = (\Psi_1 - \psi_1)\,, \tag{45}$$

then the previous equations take the form (concurrently we separate the rest energy by formal change $\epsilon = m + E$, where E stands for the nonrelativistic energy)

$$\begin{gathered} (m + E + \frac{\alpha}{r})(\Psi_1 - \psi_1) + \frac{1}{m}(\frac{d}{dr} + \frac{1}{r})\left[(\frac{d}{dr} + \frac{1}{r})(\Psi_1 + \psi_1) + \frac{i\Gamma}{r^2}(\Psi_1 - \psi_1)\right] - \frac{2\nu^2}{mr^2}(\Psi_1 + \psi_1) = m(\Psi_1 + \psi_1)\,, \\ (m + E + \frac{\alpha}{r})(\Psi_1 + \psi_1) - \frac{i\Gamma}{mr^2}\left[(\frac{d}{dr} + \frac{1}{r})(\Psi_1 + \psi_1) + \frac{i\Gamma}{r^2}(\Psi_1 - \psi_1)\right] = m(\Psi_1 - \psi_1)\,. \end{gathered}$$

Re-grouping the terms and neglecting the small component in comparison with the big one, we derive two equation (recalling that $2\nu^2 = j(j+1)$)

$$\begin{gathered} (E + \frac{\alpha}{r})\Psi_1 + \frac{1}{m}(\frac{d}{dr} + \frac{1}{r})\left[(\frac{d}{dr} + \frac{1}{r})\Psi_1 + \frac{i\Gamma}{r^2}\Psi_1\right] - \frac{j(j+1)}{mr^2}\Psi_1 = 2m\psi_1\,, \\ (E + \frac{\alpha}{r})\Psi_1 - \frac{i\Gamma}{mr^2}\left[(\frac{d}{dr} + \frac{1}{r})\Psi_1 + \frac{i\Gamma}{r^2}\Psi_1\right] = -2m\psi_1\,; \end{gathered}$$

after summing these we find a 2-nd order equation for the big component

$$\Big\{ (\frac{d}{dr}+\frac{1}{r})[(\frac{d}{dr}+\frac{1}{r})+\frac{i\Gamma}{r^2}\Psi_1]+2m(E+\frac{\alpha}{r})-\frac{j(j+1)}{r^2}-\frac{i\Gamma}{r^2}[(\frac{d}{dr}+\frac{1}{r})+\frac{i\Gamma}{r^2}\Psi_1]\Big\}\,\Psi_1=0\,.$$

Making the needed change $i\Gamma \Longrightarrow \Gamma$, we obtain

$$\Big\{ (\frac{d}{dr}+\frac{1}{r})(\frac{d}{dr}+\frac{1}{r}+\frac{\Gamma}{r^2})+2m(E+\frac{\alpha}{r})-\frac{j(j+1)}{r^2}-\frac{\Gamma}{r^2}(\frac{d}{dr}+\frac{1}{r}+\frac{\Gamma}{r^2})\Big\}\,\Psi_1=0\,.$$

The final form of the nonrelativistic radial equation is (let $\Psi_1(r)=R(r)$):

$$\Big\{ \frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+2m(E+\frac{\alpha}{r})-\frac{j(j+1)}{r^2}-\frac{2\Gamma}{r^3}-\frac{\Gamma^2}{r^4}\Big\}\,R(r)=0\,. \tag{46}$$

This equation has two singular points, $r=0$ and ∞; both of them have the rank 2. Therefore, it belongs to the class of double confluent Heun functions. In the vicinity of the point $r=0$, its Frobenius solutions are constructed in the form

$$R(r)=e^{Cr}r^{A}e^{\frac{B}{r}}f(r)\,,\qquad \frac{d^2f}{dr^2}+\left(\frac{2+2A}{r}-\frac{2B}{r^2}+2C\right)\frac{df}{dr}+$$

$$+\left(\frac{2AC+2C+2m\alpha}{r}+\frac{A^2+A-2BC-j^2-j}{r^2}+\frac{-2AB-2\Gamma}{r^3}+\frac{B^2-\Gamma^2}{r^4}+C^2+2mE\right)f=0.$$

With evident restrictions on parameters (C must be negative)

$$C=-\sqrt{-2\,mE}\,;\qquad B=\Gamma\,,\quad A=-1\,;\qquad B=-\Gamma\,,\quad A=+1 \tag{47}$$

the equation becomes simpler. Negative values of the parameter B correspond to the bound states. Depending on the sign of Γ there exist two different sets of parameters:

$$\Gamma>0,\quad A=+1,B=-\Gamma\;,\;C=-\sqrt{-2\,mE}\,;\qquad \Gamma<0,\quad A=-1,B=+\Gamma\;,\;C=-\sqrt{-2\,mE}\,. \tag{48}$$

The equation for $f(r)$ formally is the same:

$$\frac{d^2f}{dr^2}+\left(2\,C+\frac{2+2\,A}{r}-\frac{2\,B}{r^2}\right)\frac{df}{dr}+\left(\frac{2\,AC+2\,C+2\,m\alpha}{r}+\frac{A^2+A-2\,BC-j^2-j}{r^2}\right)f=0\,,$$

or shortly

$$\frac{d^2f}{dr^2}+\left(a+\frac{a_1}{r}+\frac{a_2}{r^2}\right)\frac{df}{dr}+\left(\frac{b_1}{r}+\frac{b_2}{r^2}\right)f=0\,;$$

which is equivalent to

$$r^2\frac{d^2f}{dr^2}+\left(ar^2+a_1r+a_2\right)\frac{df}{dr}+(b_1r+b_2)\,f=0\,.$$

Solutions $f(r)$ are constructed as power series:

$$f=\sum_{k=0}^{\infty}c_kr^k,\quad \frac{df}{dr}=\sum_{k=1}^{\infty}kc_kr^{k-1},\quad \frac{d^2f}{dr^2}=\sum_{k=2}^{\infty}k(k-1)c_kr^{k-2};$$

$$\sum_{k=2}^{\infty}k(k-1)c_kr^k+a\sum_{k=2}^{\infty}(k-1)c_{k-1}r^k+a_1\sum_{k=1}^{\infty}kc_kr^k+a_2\sum_{k=0}^{\infty}(k+1)c_{k+1}r^k+b_1\sum_{k=1}^{\infty}c_{k-1}r^k+b_2\sum_{k=0}^{\infty}c_kr^k=0\,.$$

Equating coefficients at the same powers of the variable r, we obtain the recurrence formulae:

$$k=0\,,\quad b_2\,c_0+a_2\,c_1=0\,,$$
$$k=1\,,\quad b_1\,c_0+(a_1+b_2)\,c_1+2\,a_2\,c_2=0\,,$$
$$k=2\,,\quad (a+b_1)\,c_1+(2+2\,a_1+b_2)\,c_2+3\,a_2\,c_3=0\,.$$

Therefore, the general formula for 3-term recurrence relations is

$$k = 1, 2,\ 3,\ 4,\ ..., \qquad [a\,(k-1) + b_1]\, c_{k-1} + [k(k-1) + a_1\, k + b_2]\, c_k + a_2(k+1)\, c_{k+1} = 0\,, \tag{49}$$

or shortly $P_{k-1}c_{k-1} + P_k c_k + P_{k+1}c_{k+1} = 0$, where

$$P_{k-1} = a\,(k-1) + b_1\,, \quad P_k = k(k-1) + a_1\, k + b_2\,, \quad P_{k+1} = a_2(k+1)\,.$$

In accordance with the Poincaré-Perrone method we divide the relation by $k^2 c_{k-1}$ and tend $k \to \infty$:

$$\frac{1}{k^2}\,[a\,(k-1) + b_1] + \frac{1}{k^2}\,[k(k-1) + a_1\, k + b_2]\,\frac{c_k}{c_{k-1}} + \frac{1}{k^2}\,a_2(k+1)\,\frac{c_{k+1}}{c_k}\,\frac{c_k}{c_{k-1}} = 0\,,$$

as a result the algebraic equation, determining possible convergence radius, is found:

$$r = 0 \quad \Rightarrow \quad R_{conv} = \frac{1}{|r|} = \infty\,. \tag{50}$$

Let us present the explicit form of the quantities entering the recurrence relations

$$P_{k-1} = 2C\,(k-1) + 2\,AC + 2\,C + 2\,m\alpha\,,$$
$$P_k = k(k-1) + (2 + 2\,A)\,k + A^2 + A - 2\,BC - j^2 - j\,, \quad P_{k+1} = -2B\,(k+1)\,,$$

and consider the transcendency condition for Heun functions:

$$P_{n-1} = 0 \quad \Longrightarrow \quad C = -\,\frac{m\alpha}{[(k-1) +\ A + 2\,]}\,, \tag{51}$$

where $C = -\sqrt{-2\,mE}$, and

$$\Gamma > 0\,, \quad A = +1, B = -\Gamma\;,\; C = -\sqrt{-2\,mE}\,; \qquad \Gamma < 0\,, \quad A = -1, B = +\Gamma\;,\; C = -\sqrt{-2\,mE}\,.$$

Whence depending on the sign of Γ, we obtain different spectra:

$$\Gamma > 0\,, \quad R(r) = e^{-\sqrt{-2mE}r}\, r e^{\frac{-\Gamma}{r}}\, f(r)\,, \quad E = -\,\frac{m\alpha^2}{2(k+2)^2}\,, \tag{52}$$

$$\Gamma < 0\,, \quad R(r) = e^{-\sqrt{-2mE}r}\, r^{-1} e^{\frac{+\Gamma}{r}}\, f(r)\,, \quad E = -\,\frac{m\alpha^2}{2k^2}\,. \tag{53}$$

The solutions of both types, respectively at $\Gamma > 0$ and $\Gamma < 0$, could describe bound states as they tend to zero at $r \to 0$. However, the formulae for energy do not depend on Γ.

VI. NONRELATIVISTIC RADIAL EQUATIONS, THE CASE OF $j = 1, 2, 3, ...$

We start from eqs. (23):

$$\begin{gathered}
-\left(\frac{d}{dr} + \frac{2}{r}\right) E_2 - 2\frac{\nu}{r} E_1 = m f_0\,, \quad +i\left(\epsilon + \frac{\alpha}{r}\right) E_1 + i\left(\frac{d}{dr} + \frac{1}{r}\right) H_1 = m f_1\,, \\
+i\left(\epsilon + \frac{\alpha}{r}\right) E_2 - 2i\frac{\nu}{r}\, H_1 = m f_2\,, \quad -i\left(\epsilon + \frac{\alpha}{r}\right) f_1 + \frac{\nu}{r} f_0 - i\frac{\Gamma}{r^2} H_1 = m E_1\,, \\
-i\left(\epsilon + \frac{\alpha}{r}\right) f_2 - \frac{d}{dr} f_0 = m E_2\,, \quad +i\,\left(\frac{d}{dr} + \frac{1}{r}\right) f_1 + i\frac{\nu}{r} f_2 - i\frac{\Gamma}{r^2} E_1 = -m H_1\,.
\end{gathered} \tag{54}$$

Excluding nondynamical variables f_0, H_1 (they do not differentiated over time) from the equations:

$$f_0 = -\frac{1}{m}\left[\left(\frac{d}{dr} + \frac{2}{r}\right) E_2 + 2\frac{\nu}{r} E_1\right], \quad H_1 = -\frac{1}{m}\left[i\left(\frac{d}{dr} + \frac{1}{r}\right) f_1 + i\frac{\nu}{r} f_2 - i\frac{\Gamma}{r^2} E_1\right].$$

we reduce remaining equations to the form

$$+i\left(\epsilon + \frac{\alpha}{r}\right) E_1 - \frac{i}{m}\left(\frac{d}{dr} + \frac{1}{r}\right)\left[i\left(\frac{d}{dr} + \frac{1}{r}\right) f_1 + i\frac{\nu}{r} f_2 - i\frac{\Gamma}{r^2} E_1\right] = m f_1\,,$$

$$+i\left(\epsilon+\frac{\alpha}{r}\right)E_2+2i\frac{\nu}{r}\frac{1}{m}\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)f_1+i\frac{\nu}{r}f_2-i\frac{\Gamma}{r^2}E_1\right]=mf_2\,;$$

$$-i\left(\epsilon+\frac{\alpha}{r}\right)f_1-\frac{\nu}{r}\frac{1}{m}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)E_2+2\frac{\nu}{r}E_1\right]+i\frac{\Gamma}{r^2}\frac{1}{m}\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)f_1+i\frac{\nu}{r}f_2-i\frac{\Gamma}{r^2}E_1\right]=mE_1\,,$$

$$-i\left(\epsilon+\frac{\alpha}{r}\right)f_2+\frac{1}{m}\frac{d}{dr}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)E_2+2\frac{\nu}{r}E_1\right]=mE_2\,.$$

Big and small components are introduced by the formulae

$$f_1=(\Psi_1+\psi_1)\,,\quad iE_1=(\Psi_1-\psi_1)\,,\quad f_2=(\Psi_2+\psi_2)\,,\quad iE_2=(\Psi_2-\psi_2)\,, \tag{55}$$

then the previous equations take the form (concurrently we separate the rest energy by the substitution $\epsilon=m+E$)

$$\left(m+E+\frac{\alpha}{r}\right)(\Psi_1-\psi_1)-\frac{i}{m}\left(\frac{d}{dr}+\frac{1}{r}\right)\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)(\Psi_1+\psi_1)+i\frac{\nu}{r}(\Psi_2+\psi_2)-\frac{\Gamma}{r^2}(\Psi_1-\psi_1)\right]=m(\Psi_1+\psi_1),$$

$$\left(m+E+\frac{\alpha}{r}\right)(\Psi_1+\psi_1)-\frac{\nu}{r}\frac{1}{m}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)(\Psi_2-\psi_2)+2\frac{\nu}{r}(\Psi_1-\psi_1)\right]-$$

$$-\frac{\Gamma}{r^2}\frac{1}{m}\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)(\Psi_1+\psi_1)+i\frac{\nu}{r}(\Psi_2+\psi_2)-\frac{\Gamma}{r^2}(\Psi_1-\psi_1)\right]=m(\Psi_1-\psi_1),$$

$$\left(m+E+\frac{\alpha}{r}\right)(\Psi_2-\psi_2)+2i\frac{\nu}{r}\frac{1}{m}\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)(\Psi_1+\psi_1)+i\frac{\nu}{r}(\Psi_2+\psi_2)-\frac{\Gamma}{r^2}(\Psi_1-\psi_1)\right]=m(\Psi_2+\psi_2),$$

$$\left(m+E+\frac{\alpha}{r}\right)(\Psi_2+\psi_2)+\frac{1}{m}\frac{d}{dr}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)(\Psi_2-\psi_2)+2\frac{\nu}{r}(\Psi_1-\psi_1)\right]=m\,(\Psi_2-\psi_2).$$

Regrouping the terms we obtain

$$\left(E+\frac{\alpha}{r}\right)(\Psi_1-\psi_1)-\frac{i}{m}\left(\frac{d}{dr}+\frac{1}{r}\right)\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)(\Psi_1+\psi_1)+i\frac{\nu}{r}(\Psi_2+\psi_2)-\frac{\Gamma}{r^2}(\Psi_1-\psi_1)\right]=2m\psi_1,$$

$$\left(E+\frac{\alpha}{r}\right)(\Psi_1+\psi_1)-\frac{\nu}{r}\frac{1}{m}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)(\Psi_2-\psi_2)+2\frac{\nu}{r}(\Psi_1-\psi_1)\right]-$$

$$-\frac{\Gamma}{r^2}\frac{1}{m}\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)(\Psi_1+\psi_1)+i\frac{\nu}{r}f_2-\frac{\Gamma}{r^2}(\Psi_1-\psi_1)\right]=-2m\psi_1,$$

$$\left(E+\frac{\alpha}{r}\right)(\Psi_2-\psi_2)+2i\frac{\nu}{r}\frac{1}{m}\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)(\Psi_1+\psi_1)+i\frac{\nu}{r}(\Psi_2+\psi_2)-\frac{\Gamma}{r^2}(\Psi_1-\psi_1)\right]=2m\psi_2,$$

$$\left(E+\frac{\alpha}{r}\right)(\Psi_2+\psi_2)+\frac{1}{m}\frac{d}{dr}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)(\Psi_2-\psi_2)+2\frac{\nu}{r}(\Psi_1-\psi_1)\right]=-2m\psi_2.$$

To derive needed equations for big components Ψ_1 and Ψ_2, we sum the equations in each pair and then neglect small components in comparison with the big ones. This results in

$$2\left(E+\frac{\alpha}{r}\right)\Psi_1+\frac{1}{m}\left(\frac{d}{dr}+\frac{1}{r}\right)\left[\left(\frac{d}{dr}+\frac{1}{r}\right)\Psi_1+\frac{\nu}{r}\Psi_2+i\frac{\Gamma}{r^2}\Psi_1\right]-$$

$$-\frac{\nu}{r}\frac{1}{m}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)\Psi_2+2\frac{\nu}{r}\Psi_1\right]-\frac{i\Gamma}{r^2}\frac{1}{m}\left[\left(\frac{d}{dr}+\frac{1}{r}\right)\Psi_1+\frac{\nu}{r}\Psi_2+\frac{i\Gamma}{r^2}\Psi_1\right]=0\,,$$

$$2\left(E+\frac{\alpha}{r}\right)\Psi_2-2\frac{\nu}{r}\frac{1}{m}\left[\left(\frac{d}{dr}+\frac{1}{r}\right)\Psi_1+\frac{\nu}{r}\Psi_2+\frac{i\Gamma}{r^2}\Psi_1\right]+\frac{1}{m}\frac{d}{dr}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)\Psi_2+2\frac{\nu}{r}\Psi_1\right]=0\,.$$

Allowing for that Γ is imaginary, we make the change $i\Gamma \Longrightarrow \Gamma$, so producing the system

$$2m\left(E+\frac{\alpha}{r}\right)\Psi_1+\left(\frac{d}{dr}+\frac{1}{r}\right)\left[\left(\frac{d}{dr}+\frac{1}{r}\right)\Psi_1+\frac{\nu}{r}\Psi_2+\frac{\Gamma}{r^2}\Psi_1\right]-$$

$$-\frac{\nu}{r}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)\Psi_2+2\frac{\nu}{r}\Psi_1\right]-\frac{\Gamma}{r^2}\left[\left(\frac{d}{dr}+\frac{1}{r}\right)\Psi_1+\frac{\nu}{r}\Psi_2+\frac{\Gamma}{r^2}\Psi_1\right]=0\,,$$

$$2\,m\left(E+\frac{\alpha}{r}\right)\Psi_2-2\,\frac{\nu}{r}\left[\left(\frac{d}{dr}+\frac{1}{r}\right)\Psi_1+\frac{\nu}{r}\Psi_2+\frac{\Gamma}{r^2}\Psi_1\right]+\frac{d}{dr}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)\Psi_2+2\frac{\nu}{r}\Psi_1\right]=0\,.$$

Further we get (recalling that $2\nu^2=j(j+1)$)

$$\begin{gathered}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+2mE+\frac{2m\alpha}{r}-\frac{2\,\nu^2}{r^2}-\frac{2\,\Gamma}{r^3}-\frac{\Gamma^2}{r^4}\right)\Psi_1-\nu\left(\frac{2}{r^2}+\frac{\Gamma}{r^3}\right)\Psi_2=0\,,\\ \left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+2mE+\frac{2m\alpha}{r}-\frac{2\nu^2+2}{r^2}\right)\Psi_2-2\,\nu\left(\frac{2}{r^2}+\frac{\Gamma}{r^3}\right)\Psi_1=0\,.\end{gathered} \tag{56}$$

This system for two functions permits us to construct the fourth order equations for functions $\Psi_1(r)$ and $\Psi_2(r)$. It suffices to study only one of them, for example, for the function Ψ_1. We will use dimensionless variables:

$$x=rm\,,\quad \Gamma m=\gamma\,,\quad \epsilon=\frac{E}{m}\,; \tag{57}$$

as a result we get

$$\begin{gathered}\frac{d^4\Psi_1}{dx^4}+\left[\frac{10}{x}-\frac{4}{2\,x+\gamma}\right]\frac{d^3\Psi_1}{dx^3}+\\ +\left[4\,\epsilon+\frac{-24+4\,\alpha\,\gamma}{\gamma x}+\frac{22-4\,\nu^2}{x^2}-\frac{2\,\gamma}{x^3}-\frac{\gamma^2}{x^4}+\frac{48}{(2\,x+\gamma)\,\gamma}+\frac{8}{(2\,x+\gamma)^2}\right]\frac{d^2\Psi_1}{dx^2}+\\ +\left[\frac{64-16\,\nu^2+20\,\gamma^2\epsilon-8\,\alpha\,\gamma}{\gamma^2x}+\frac{-24+8\,\nu^2+16\,\alpha\,\gamma}{\gamma x^2}+\frac{8-12\,\nu^2}{x^3}+\right.\\ \left.+\frac{-128-8\,\gamma^2\epsilon+16\,\alpha\,\gamma+32\,\nu^2}{(2\,x+\gamma)\,\gamma^2}-\frac{32}{(2\,x+\gamma)^2\,\gamma}\right]\frac{d\Psi_1}{dx}+\\ +\left[+4\epsilon^2+\frac{128\nu^2+8\epsilon\alpha\gamma^3+64\alpha\gamma-32\gamma^2\epsilon}{x\gamma^3}+\frac{-24\,\alpha\gamma-48\nu^2+20\gamma^2\epsilon+4\alpha^2\gamma^2-8\epsilon\nu^2\gamma^2}{\gamma^2x^2}+\right.\\ +\frac{8\,\alpha\,\gamma-4\,\gamma^2\epsilon+16\,\nu^2-8\,\alpha\,\nu^2\gamma}{\gamma x^3}+\frac{-8\,\nu^2-4\,\alpha\,\gamma-2\,\gamma^2\epsilon+4\,\nu^4}{x^4}-\\ \left.-\frac{2\,\gamma\left(-2+2\nu^2+\alpha\gamma\right)}{x^5}+\frac{2\gamma^2}{x^6}+\frac{-128\alpha\gamma+64\gamma^2\epsilon-256\nu^2}{(2x+\gamma)\,\gamma^3}+\frac{-32\alpha\gamma+16\gamma^2\epsilon-64\nu^2}{(2x+\gamma)^2\,\gamma^2}\right]\Psi_1=0\,.\end{gathered} \tag{58}$$

Symbolic structure of the equation (58) is written as follows (let $|si_1 = \Psi$)

$$\frac{d^4}{dx^4}\Psi_1 + \left[\frac{10}{x} - \frac{4}{2x+\gamma}\right]\frac{d^3}{dx^3}\Psi + \left[4\epsilon + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \frac{a_4}{x^4} + \frac{a_5}{2x+\gamma} + \frac{a_6}{(2x+\gamma)^2}\right]\frac{d^2}{dx^2}\Psi +$$
$$+\left[\frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \frac{b_4}{2x+\gamma} + \frac{b_5}{(2x+\gamma)^2}\right]\frac{d}{dx}\Psi +$$
$$+\left[4\epsilon^2 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \frac{c_4}{x^4} + \frac{c_5}{x^5} + \frac{c_6}{x^6} + \frac{c_7}{2x+\gamma} + \frac{c_8}{(2x+\gamma)^2}\right]\Psi = 0\,. \quad (59)$$

We will search Frobenius type solutions in the form

$$\Psi(x) = x^A e^{Bx} e^{C/x} f(x)\,, \quad (60)$$

this yields

$$\frac{d^4 f}{dx^4} f + \left[\frac{4A+10}{x} - \frac{4C}{x^2} + 4B - \frac{4}{(2x+\gamma)}\right]\frac{d^3 f}{dx^3} +$$
$$+\left[\frac{-24C + a_1\gamma^2 - 12A\gamma + 30B\gamma^2 + 12AB\gamma^2}{\gamma^2 x} + \frac{6A^2\gamma + a_2\gamma + 12C - 12BC\gamma + 24A\gamma}{\gamma x^2} +\right.$$
$$\left.+\frac{a_3 - 12AC - 18C}{x^3} + \frac{6C^2 + a_4}{x^4} + \frac{48C + 24A\gamma - 12B\gamma^2 + a_5\gamma^2}{(2x+\gamma)\,\gamma^2} + 6B^2 + 4\epsilon + \frac{a_6}{(2x+\gamma)^2}\right]\frac{d^2 f}{dx^2} +$$
$$+\left[\frac{1}{\gamma^4 x}\left(8\epsilon A\gamma^4 + 2a_1 B\gamma^4 + 24A^2\gamma^2 - 24A\gamma^2 - 96C\gamma + 96C^2 + 12AB^2\gamma^4 + 2a_5\, A\gamma^3 +\right.\right.$$
$$\left.+4a_5 C\gamma^2 + 2a_6 A\gamma^2 + 8a_6 C\gamma - 24AB\gamma^3 + 96AC\gamma + 30B^2\gamma^4 + b_1\gamma^4 - 48\,BC\gamma^2\right) +$$
$$+\frac{1}{\gamma^3 x^2}\left(-8\epsilon C\gamma^3 + 2a_1 A\gamma^3 + 2a_2 B\gamma^3 - 12A^2\gamma^2 + 12A\gamma^2 + 48C\gamma - 48C^2 +\right.$$
$$\left.+48AB\gamma^3 + 12A^2B\gamma^3 - 12B^2C\gamma^3 - 2a_5 C\gamma^2 - 2a_6 C\gamma - 48AC\gamma + b_2\gamma^3 + 24BC\gamma^2\right) +$$
$$+\frac{1}{\gamma^2 x^3}\left(-2a_1 C\gamma^2 + 2a_2 A\gamma^2 + 2a_3 B\gamma^2 - 24C\gamma + 24C^2 - 36BC\gamma^2 +\right.$$
$$\left.+24AC\gamma + 4A^3\gamma^2 + 18A^2\gamma^2 - 22A\gamma^2 + b_3\gamma^2 - 24ABC\gamma^2\right) +$$
$$+\frac{-2a_2 C\gamma + 2a_3 A\gamma + 2a_4 B\gamma - 12C^2 - 24AC\gamma + 12BC^2\gamma - 12A^2C\gamma + 36C\gamma}{\gamma x^4} +$$
$$+\frac{-2a_3 C + 2a_4 A + 12AC^2 + 6C^2}{x^5} - \frac{2C\left(a_4 + 2C^2\right)}{x^6} + 8\epsilon B + 4B^3 + \frac{2a_6 B\gamma^2 - 4a_6 A\gamma - 8a_6 C + b_5\gamma^2}{(2x+\gamma)^2\gamma^2} +$$
$$+\frac{1}{(2\,x+\gamma)\,\gamma^4}\left(2\,a_5\,B\gamma^4 - 48A^2\gamma^2 + 48A\gamma^2 + 192C\gamma - 192C^2 - 4a_5 A\gamma^3 -\right.$$
$$\left.\left.-8a_5 C\gamma^2 - 4a_6 A\gamma^2 - 16a_6 C\gamma + 48AB\gamma^3 - 192\,AC\gamma - 12B^2\gamma^4 + b_4\gamma^4 + 96BC\gamma^2\right)\right]\frac{df}{dx} +$$

$$+\left[\frac{1}{\gamma^6 x}\left(288AC\gamma^2+384C^2\gamma-96A^2C\gamma^2+96BC^2\gamma^2-32A\gamma^3-16A^3\gamma^3-192AC^2\gamma-\right.\right.$$

$$-8a_5C^2\gamma^2-96BC\gamma^3-2a_5A^2\gamma^4+2a_5A\gamma^4+24a_6C\gamma^2+2b_4C\gamma^4+8a_5C\gamma^3+4a_6A\gamma^3-$$

$$-24AB\gamma^4-24B^2C\gamma^4-32a_6C^2\gamma+24A^2B\gamma^4-4a_6A^2\gamma^3+4b_5C\gamma^3-8a_5AC\gamma^3+96ABC\gamma^3+$$

$$+4a_5BC\gamma^4-24a_6AC\gamma^2+8a_6BC\gamma^3+2a_5AB\gamma^5+2a_6AB\gamma^4-128C^3-12AB^2\gamma^5+48A^2\gamma^3-$$

$$\left.-192C\gamma^2+10B^3\gamma^6+b_1B\gamma^6+a_1B^2\gamma^6+4AB^3\gamma^6+b_4A\gamma^5+b_5A\gamma^4+c_1\gamma^6+8\epsilon AB\gamma^6\right)+$$

$$+\frac{1}{\gamma^5x^2}\left(-144AC\gamma^2-192C^2\gamma+48A^2C\gamma^2-48BC^2\gamma^2+b_2B\gamma^5+16A\gamma^3+8A^3\gamma^3-4B^3C\gamma^5+\right.$$

$$+b_1A\gamma^5+96AC^2\gamma+2a_1AB\gamma^5-8\epsilon BC\gamma^5+4a_5C^2\gamma^2+48BC\gamma^3+a_5A^2\gamma^4-a_5A\gamma^4-$$

$$-8a_6C\gamma^2-b_4C\gamma^4-4a_5C\gamma^3-a_6A\gamma^3+12AB\gamma^4+12B^2C\gamma^4+12a_6C^2\gamma-12A^2B\gamma^4+$$

$$+a_6\,A^2\gamma^3-b_5C\gamma^3+4a_5AC\gamma^3-48ABC\gamma^3-2a_5BC\gamma^4+8a_6AC\gamma^2-2a_6BC\gamma^3+64C^3+$$

$$\left.+24AB^2\gamma^5-24A^2\gamma^3+a_2B^2\gamma^5+4\epsilon A^2\gamma^5+6A^2B^2\gamma^5-4\epsilon A\gamma^5+96C\gamma^2+c_2\gamma^5\right)+$$

$$+\frac{1}{\gamma^4x^3}\left(72AC\gamma^2+96C^2\gamma-24A^2C\gamma^2+24BC^2\gamma^2-8A\gamma^3-4A^3\gamma^3-48AC^2\gamma-2a_5C^2\gamma^2-\right.$$

$$-24BC\gamma^3+2a_6C\gamma^2+2a_5C\gamma^3-22AB\gamma^4-18B^2C\gamma^4-4a_6C^2\gamma+18A^2B\gamma^4-2a_1BC\gamma^4+$$

$$+2a_2AB\gamma^4-8\epsilon AC\gamma^4-12AB^2C\gamma^4-2a_5AC\gamma^3+24ABC\gamma^3-2a_6AC\gamma^2-32C^3+12A^2\gamma^3-$$

$$\left.-48C\gamma^2+c_3\gamma^4+4A^3B\gamma^4+b_2A\gamma^4+a_3B^2\gamma^4-a_1A\gamma^4+8\epsilon C\gamma^4+a_1A^2\gamma^4+b_3B\gamma^4-b_1\,C\gamma^4\right)+$$

$$+\frac{1}{\gamma^3x^4}\left(-36\,AC\gamma^2-48C^2\gamma+12A^2C\gamma^2-12BC^2\gamma^2+14A\gamma^3+4A^3\gamma^3+24AC^2\gamma+\right.$$

$$+a_5C^2\gamma^2+36BC\gamma^3+a_6C^2\gamma-12A^2BC\gamma^3-2a_1AC\gamma^3+2a_3AB\gamma^3-2a_2BC\gamma^3-$$

$$-24ABC\gamma^3+16C^3-19A^2\gamma^3+24C\gamma^2+c_4\gamma^3+A^4\gamma^3-a_2A\gamma^3+2a_1\,C\gamma^3+$$

$$\left.+a_4B^2\gamma^3+a_2A^2\gamma^3+4\epsilon C^2\gamma^3-b_2C\gamma^3+6B^2C^2\gamma^3+b_3A\gamma^3\right)+$$

$$+\frac{1}{\gamma^2x^5}\left(46AC\gamma^2+a_1C^2\gamma^2+a_3A^2\gamma^2+2a_2C\gamma^2-a_3A\gamma^2-b_3C\gamma^2+24C^2\gamma-8C^3-6A^2C\gamma^2+\right.$$

$$\left.+6BC^2\gamma^2-4A^3C\gamma^2-2a_3BC\gamma^2+2a_4AB\gamma^2-12AC^2\gamma-36C\gamma^2+c_5\gamma^2+12ABC^2\gamma^2-2a_2AC\gamma^2\right)+$$

$$+\frac{a_2C^2\gamma + a_4A^2\gamma + 2a_3C\gamma - a_4A\gamma + 4C^3 + 6A^2C^2\gamma - 4BC^3\gamma - 2a_3AC\gamma - 2a_4BC\gamma - 24C^2\gamma + c_6\gamma}{\gamma x^6} -$$

$$-\frac{C\left(-a_3C - 2a_4 + 4AC^2 + 2a_4A - 2C^2\right)}{x^7} + \frac{C^2\left(a_4 + C^2\right)}{x^8} + B^4 + 4\,\epsilon^2 + 4\epsilon B^2 +$$

$$+\frac{1}{(2x+\gamma)\,\gamma^6}\left(-576AC\gamma^2 - 768C^2\gamma + 192A^2C\gamma^2 - 192BC^2\gamma^2 + 64A\gamma^3 + 32A^3\gamma^3 + 384AC^2\gamma +\right.$$

$$+16a_5C^2\gamma^2 + 192BC\gamma^3 + 4a_5A^2\gamma^4 - 4a_5A\gamma^4 - 48a_6C\gamma^2 - 4b_4C\gamma^4 - 16a_5C\gamma^3 - 8a_6A\gamma^3 +$$

$$+48AB\gamma^4 + 48B^2C\gamma^4 + 64a_6C^2\gamma - 48A^2B\gamma^4 + 8a_6A^2\gamma^3 - 8b_5\gamma^3 + 16a_5AC\gamma^3 - 192ABC\gamma^3 -$$

$$-8a_5BC\gamma^4 + 48a_6AC\gamma^2 - 16a_6BC\gamma^3 - 4a_5AB\gamma^5 - 4a_6AB\gamma^4 + 256C^3 +$$

$$\left.+24AB^2\gamma^5 - 96A^2\gamma^3 + 384C\gamma^2 - 4B^3\gamma^6 + c_7\gamma^6 + a_5B^2\gamma^6 + b_4B\gamma^6 - 2b_4A\gamma^5 - 2b_5\,A\gamma^4\right) +$$

$$+\frac{1}{(2x+\gamma)^2\gamma^4}\left(a_6B^2\gamma^4 + b_5B\gamma^4 + 4a_6A^2\gamma^2 + 16a_6C^2 - 2b_5\gamma^3 - 4a_6A\gamma^2 -\right.$$
$$\left.\left.-16a_6C\gamma - 4b_5C\gamma^2 + c_8\gamma^4 - 4a_6AB\gamma^3 + 16a_6AC\gamma - 8a_6BC\gamma^2\right)\right] f = 0\,. \tag{61}$$

Impose restrictions on parameters B and C:

$$B^4 + 4\,\epsilon^2 + 4\,\epsilon\,B^2 = 0 \quad\Longrightarrow\quad B = -\sqrt{-2\,\epsilon}, +\sqrt{-2\,\epsilon}\,, \tag{62}$$

$$\frac{1}{x^8},\quad C^2\left(a_4 + C^2\right) = 0 \;\Rightarrow\; C = 0,\; \pm\sqrt{-a_4} \;\Rightarrow\; C_1 = 0\,,\; C_2 = +\gamma\,,\; C_3 = -\gamma\,. \tag{63}$$

While $C = C_1 = 0$ the coefficient at $1/x^7$ vanishes, so we require the coefficient at $1/x^6$ be equal to zero as well:

$$a_2C^2\gamma + a_4A^2\gamma + 2a_3C\gamma - a_4A\gamma + 4\,C^3 + 6A^2C^2\gamma - 4BC^3\gamma - 2\,a_3AC\gamma - 2a_4BC\gamma - 24C^2\gamma + c_6\gamma = 0\,,$$

or

$$a_4\,A^2\gamma - a_4\,A\gamma + c_6\,\gamma = 0 \;\Longrightarrow\; -\gamma^3A^2 + \gamma^3A + 2\,\gamma^3 = 0 \;\Longrightarrow\; A_1 = -1,\; A_2 = +2\,. \tag{64}$$

If $C = C_2 = +\gamma$, we demand the multiplier at $1/x^7$ be equal to zero:

$$-a_3\,C - 2\,a_4 + 4\,AC^2 + 2\,a_4\,A - 2\,C^2 = 0\,,\; a_3 = -2\gamma \;\Longrightarrow\; 2\,\gamma^2 + 2\,A\gamma^2 = 0 \quad\Longrightarrow\quad A_3 = -1\,. \tag{65}$$

Let $C = C_3 = -\gamma$, then we demand the multiplier at $1/x^7$ be equal to zero:

$$-a_3\,C - 2\,a_4 + 4\,AC^2 + 2\,a_4\,A - 2\,C^2 = 0\,,\quad a_3 = -2\gamma \;\Longrightarrow\; -2\,\gamma^2 + 2\,A\gamma^2 = 0 \;\Longrightarrow\; A_4 = +1\,. \tag{66}$$

Thus, there are four types of the solutions (below only negative values of B are used):

$$\begin{aligned}
I,\quad B = -\sqrt{-2\epsilon},\quad C = 0,\quad A = -1,\quad \Psi = e^{Bx}\frac{1}{x}f_1(x);\\
II,\quad B = -\sqrt{-2\epsilon},\quad C = 0,\quad A = +2,\quad \Psi = e^{Bx}x^2 f_2(x)\,;\\
III,\quad B = -\sqrt{-2\epsilon},\quad C = +\gamma,\quad A = -1,\quad \Psi(x) = e^{Bx}\frac{1}{x}e^{+\gamma/x}f_3(x)\,;\\
IV,\quad B = -\sqrt{-2\epsilon},\quad C_3 = -\gamma,\quad A = +1,\quad \Psi(x) = e^{Bx}xe^{-\gamma/x}f_4(x)\,.
\end{aligned} \tag{67}$$

Only three cases may be be suitable for describing the bound states:

$$II\,; \quad III \quad \text{at negative } \gamma\,; \quad IV \qquad \text{at positive } \gamma\,. \tag{68}$$

Let us examine the variant II. The main equation takes the following form:

$$\begin{aligned}
&\frac{d^4\Phi}{dx^4} + \left[4\,B + \frac{18}{x} - \frac{4}{2\,x+\gamma}\right]\frac{d^3\Phi}{dx^3} + \\
&+\left[4\epsilon + 6B^2 + \frac{a_1\gamma + 54\,B\gamma - 24}{\gamma x} + \frac{a_2+72}{x^2} + \frac{a_3}{x^3} + \frac{a_4}{x^4} + \frac{a_5\gamma - 12\,B\gamma + 48}{(2x+\gamma)\,\gamma} + \frac{a_6}{(2x+\gamma)^2}\right]\frac{d^2\Phi}{dx^2} + \\
&+\left[\frac{4\,a_5\,\gamma + 48 + b_1\,\gamma^2 + 16\,\epsilon\,\gamma^2 + 4\,a_6 - 48\,B\gamma + 2\,a_1\,B\gamma^2 + 54\,B^2\gamma^2}{\gamma^2 x}+\right.\\
&+\frac{-24 + 4\,a_1\,\gamma + b_2\,\gamma + 2\,a_2\,B\gamma + 144\,B\gamma}{\gamma x^2} + \frac{b_3 + 60 + 4\,a_2 + 2\,a_3\,B}{x^3} + \frac{4\,a_3 + 2\,a_4\,B}{x^4} + \frac{4\,a_4}{x^5} + \\
&\left.+\frac{-96 + b_4\,\gamma^2 - 8\,a_6 - 8\,a_5\,\gamma + 96\,B\gamma + 2\,a_5\,B\gamma^2 - 12\,B^2\gamma^2}{(2\,x+\gamma)\,\gamma^2} + \frac{b_5\,\gamma - 8\,a_6 + 2\,a_6\,B\gamma}{(2\,x+\gamma)^2\,\gamma}\right]\frac{d\Phi}{dx} + \\
&+\left[\frac{1}{\gamma^3 x}\left(4\,a_5\,B\gamma^2 + 4\,a_6\,B\gamma + 2\,b_4\,\gamma^2 - 4\,a_5\,\gamma - 8\,a_6 +\right.\right.\\
&\left.+2\,b_5\,\gamma + c_1\,\gamma^3 + a_1\,B^2\gamma^3 + b_1\,B\gamma^3 - 24\,B^2\gamma^2 + 48\,B\gamma + 16\,\epsilon\,B\gamma^3 + 18\,B^3\gamma^3\right) + \\
&+\frac{8\,\epsilon\,\gamma^2 + 2\,a_5\,\gamma + c_2\,\gamma^2 + 2\,b_1\,\gamma^2 + 2\,a_6 + a_2\,B^2\gamma^2 + b_2\,B\gamma^2 - 24\,B\gamma + 4\,a_1\,B\gamma^2 + 72\,B^2\gamma^2}{\gamma^2 x^2} + \\
&+\frac{2\,a_1 + 2\,b_2 + c_3 + a_3\,B^2 + b_3\,B + 4\,a_2\,B + 60\,B}{x^3} + \frac{c_4 + 2\,a_2 + 2\,b_3 + 4\,a_3\,B + a_4\,B^2}{x^4} + \\
&+\frac{c_5 + 2\,a_3 + 4\,a_4\,B}{x^5} + \frac{1}{(2\,x+\gamma)\,\gamma^3}\left(-8\,a_5\,B\gamma^2 - 8\,a_6\,B\gamma + 8\,a_5\,\gamma + c_7\,\gamma^3 - 4\,b_5\,\gamma - 4\,b_4\,\gamma^2\right.\\
&\left.+16\,a_6 + a_5\,B^2\gamma^3 + b_4\,B\gamma^3 + 48\,B^2\gamma^2 - 4\,B^3\gamma^3 - 96\,B\gamma\right) + \\
&\left.+\frac{-8\,a_6\,B\gamma + c_8\,\gamma^2 - 4\,b_5\,\gamma + 8\,a_6 + a_6\,B^2\gamma^2 + b_5\,B\gamma^2}{(2\,x+\gamma)^2\,\gamma^2}\right]\Phi = 0\,.
\end{aligned} \tag{69}$$

Shortly this equation can be written t as

$$\begin{aligned}
&\frac{d^4\Phi}{dx^4} + \left[P_0 + \frac{P_1}{x} + \frac{p_1}{2x+\gamma}\right]\frac{d^3\Phi}{dx^3} + \left[Q_0 + \frac{Q_1}{x} + \frac{Q_2}{x^2} + \frac{Q_3}{x^3} + \frac{Q_4}{x^4} + \frac{q_1}{2x+\gamma} + \frac{q_2}{(2x+\gamma)^2}\right]\frac{d^2\Phi}{dx^2} + \\
&+\left[\frac{M_1}{x} + \frac{M_2}{x^2} + \frac{M_3}{x^3} + \frac{M_4}{x^4} + \frac{M_5}{x^5} + \frac{m_1}{2x+\gamma} + \frac{m_2}{(2x+\gamma)^2}\right]\frac{d\Phi}{dx} + \\
&+\left[\frac{N_1}{x} + \frac{N_2}{x^2} + \frac{N_3}{x^3} + \frac{N_4}{x^4} + \frac{N_5}{x^5} + \frac{n_1}{2x+\gamma} + \frac{n_2}{(2x+\gamma)^2}\right]\Phi = 0\,.
\end{aligned} \tag{70}$$

Let us multiply eq. (70) by $x^5(2x+\gamma)^2$:

$$\begin{aligned}
&\left[4\,x^7 + 4\,\gamma x^6 + \gamma^2 x^5\right]\frac{d^4\Phi}{dx^4} + \left[4\,P_0\,x^7 + (4\,P_0\,\gamma + 4\,P_1 + 2\,p_1)\,x^6 + \left(P_0\,\gamma^2 + 4\,P_1\,\gamma + p_1\,\gamma\right)x^5 + P_1\,\gamma^2 x^4\right]\frac{d^3\Phi}{dx^3} + \\
&+\left[4\,Q_0\,x^7 + (4\,Q_0\,\gamma + 4\,Q_1 + 2\,q_1)\,x^6 + \left(Q_0\,\gamma^2 + 4\,Q_1\,\gamma + 4\,Q_2 + \gamma q_1 + q_2\right)x^5 +\right.\\
&\left.+\left(Q_1\gamma^2 + 4Q_2\gamma + 4Q_3\right)x^4 + \left(Q_2\gamma^2 + 4Q_3\gamma + 4Q_4\right)x^3 + \left(Q_3\gamma^2 + 4Q_4\gamma\right)x^2 + Q_4\gamma^2 x\right]\frac{d^2\Phi}{dx^2} + \\
&+\left[(4\,M_1 + 2\,m_1)\,x^6 + (4\,M_1\,\gamma + 4\,M_2 + m_1\,\gamma + m_2)\,x^5 + \left(M_1\,\gamma^2 + 4\,M_2\,\gamma + 4\,M_3\right)x^4 +\right.\\
&\left.+\left(M_2\gamma^2 + 4M_3\gamma + 4\,M_4\right)x^3 + \left(M_3\gamma^2 + 4M_4\gamma + 4\,M_5\right)x^2 + \left(M_4\gamma^2 + 4M_5\gamma\right)x + M_5\gamma^2\right]\frac{d\Phi}{dx} + \\
&+\left[(4\,N_1 + 2\,n_1)\,x^6 + (4\,N_1\,\gamma + 4\,N_2 + \gamma n_1 + n_2)\,x^5 + \left(N_1\,\gamma^2 + 4\,N_2\,\gamma + 4\,N_3\right)x^4 +\right.\\
&\left.+\left(N_2\gamma^2 + 4N_3\gamma + 4N_4\right)x^3 + \left(N_3\gamma^2 + 4N_4\gamma + 4N_5\right)x^2 + \left(N_4\gamma^2 + 4N_5\gamma\right)x + N_5\gamma^2\right]\Phi = 0\,,
\end{aligned}$$

and search its solution as the power series: $\Phi = \sum_{l=0}^{\infty} d_l x^l$ After calculation we we find to 8-term recurrence relations ($k \geq 6$):

$$
\begin{aligned}
(4N_1 + 2n_1)\, d_{k-6} + [4Q_0(k-5)(k-6)\,(4M_1 + 2m_1)\,(k-5) + (4N_1\gamma + 4\,N_2 + \gamma n_1 + n_2)]\, d_{k-5} + \\
+ [4\,P_0\,(k-4)(k-5)(k-6) + (4\,Q_0\,\gamma + 4\,Q_1 + 2\,q_1)\;(k-4)(k-5) + \\
+ (4\,M_1\,\gamma + 4\,M_2 + m_1\,\gamma + m_2)\;(k-4) + \big(N_1\,\gamma^2 + 4\,N_2\,\gamma + 4\,N_3\big)\big]\; d_{k-4} + \\
+ [4\,(k-3)(k-4)(k-5)(k-6) + (4\,P_0\,\gamma + 4\,P_1 + 2\,p_1)\;(k-3)(k-4)(k-5) + \\
+ \big(Q_0\,\gamma^2 + 4\,Q_1\,\gamma + 4\,Q_2 + \gamma q_1 + q_2\big)\;(k-3)(k-4) + \\
+ \big(M_1\,\gamma^2 + 4\,M_2\,\gamma + 4\,M_3\big)\;(k-3) + \big(N_2\,\gamma^2 + 4\,N_3\,\gamma + 4\,N_4\big)\big]\; d_{k-3} + \\
+ \big[4\,\gamma(k-2)(k-3)(k-4)(k-5) + \big(P_0\,\gamma^2 + 4\,P_1\,\gamma + p_1\,\gamma\big)\;(k-2)(k-3)(k-4) + \\
+ \big(Q_1\,\gamma^2 + 4\,Q_2\,\gamma + 4\,Q_3\big)\;(k-2)(k-3) + \\
+ \big(M_2\,\gamma^2 + 4\,M_3\,\gamma + 4\,M_4\big)\;(k-2) + \big(N_3\,\gamma^2 + 4\,N_4\,\gamma + 4\,N_5\big)\big]\; d_{k-2} + \\
+ \big[\gamma^2\,(k-1)(k-2)(k-3)(k-4) + P_1\,\gamma^2\,(k-1)(k-2)(k-3) + \\
+ \big(Q_2\gamma^2 + 4Q_3\gamma + 4Q_4\big)\,(k-1)(k-2) + \big(M_3\gamma^2 + 4M_4\gamma + 4M_5\big)\,(k-1) + \\
+ \big(N_4\gamma^2 + 4N_5\gamma\big)\big]\, d_{k-1} + + \big[\big(Q_3\gamma^2 + 4Q_4\gamma\big)\,k(k-1) + \big(M_4\gamma^2 + 4M_5\gamma\big)\,k + N_5\gamma^2\big]\, d_k + \\
+ \big[Q_4\,\gamma^2\,(k+1)\,k + M_5\,\gamma^2\,(k+1)\big]\, d_{k+1} = 0\,,
\end{aligned}
\tag{71}
$$

or shortly

$$
\begin{aligned}
D_{k-6}d_{k-6} + D_{k-5}d_{k-5} + D_{k-4}d_{k-4} + D_{k-3}d_{k-3} + \\
+ D_{k-2}d_{k-2} + D_{k-1}d_{k-1} + D_k d_k + D_{k+1}d_{k+1} = 0\,.
\end{aligned}
\tag{72}
$$

We divide the eq. (71) by $d_{k-6}k^4$:

$$
\begin{aligned}
[4N_1 + 2n_1] + [4Q_0(k-5)(k-6) + (4M_1 + 2m_1)\,(k-5) + (4N_1\gamma + 4N_2 + \gamma n_1 + n_2)]\,\frac{d_{k-5}}{d_{k-6}} + \\
+ [4\,P_0\,(k-4)(k-5)(k-6) + (4\,Q_0\,\gamma + 4\,Q_1 + 2\,q_1)\;(k-4)(k-5) + \\
+ (4\,M_1\,\gamma + 4\,M_2 + m_1\,\gamma + m_2)\;(k-4) + \big(N_1\,\gamma^2 + 4\,N_2\,\gamma + 4\,N_3\big)\big]\;\frac{d_{k-4}}{d_{k-5}}\frac{d_{k-5}}{d_{k-6}} + \\
+ [4\,(k-3)(k-4)(k-5)(k-6) + (4\,P_0\,\gamma + 4\,P_1 + 2\,p_1)\;(k-3)(k-4)(k-5) + \\
+ \big(Q_0\,\gamma^2 + 4\,Q_1\,\gamma + 4\,Q_2 + \gamma q_1 + q_2\big)\;(k-3)(k-4) + \\
+ \big(M_1\,\gamma^2 + 4\,M_2\,\gamma + 4\,M_3\big)\;(k-3) + \big(N_2\,\gamma^2 + 4\,N_3\,\gamma + 4\,N_4\big)\big]\;\frac{d_{k-3}}{d_{k-4}}\frac{d_{k-4}}{d_{k-5}}\frac{d_{k-5}}{d_{k-6}} + \\
+ \big[4\,\gamma(k-2)(k-3)(k-4)(k-5) + \big(P_0\,\gamma^2 + 4\,P_1\,\gamma + p_1\,\gamma\big)\;(k-2)(k-3)(k-4) + \\
+ \big(Q_1\,\gamma^2 + 4\,Q_2\,\gamma + 4\,Q_3\big)\;(k-2)(k-3) + \\
+ \big(M_2\,\gamma^2 + 4\,M_3\,\gamma + 4\,M_4\big)\;(k-2) + \big(N_3\,\gamma^2 + 4\,N_4\,\gamma + 4\,N_5\big)\big]\;\frac{d_{k-2}}{d_{k-3}}\frac{d_{k-3}}{d_{k-4}}\frac{d_{k-4}}{d_{k-5}}\frac{d_{k-5}}{d_{k-6}} + \\
+ \big[\gamma^2(k-1)(k-2)(k-3)(k-4) + P_1\gamma^2(k-1)(k-2)(k-3) + \\
+ \big(Q_2\gamma^2 + 4Q_3\gamma + 4Q_4\big)\,(k-1)(k-2) +
\end{aligned}
$$

$$
\begin{aligned}
+ \big(M_3\,\gamma^2 + 4\,M_4\,\gamma + 4\,M_5\big)\;(k-1) + \big(N_4\,\gamma^2 + 4\,N_5\,\gamma\big)\big]\;\frac{d_{k-1}}{d_{k-2}}\frac{d_{k-2}}{d_{k-3}}\frac{d_{k-3}}{d_{k-4}}\frac{d_{k-4}}{d_{k-5}}\frac{d_{k-5}}{d_{k-6}} + \\
+ \big[\big(Q_3\gamma^2 + 4Q_4\gamma\big)\,k(k-1) + \big(M_4\gamma^2 + 4M_5\gamma\big)\,k + N_5\gamma^2\big]\;\frac{d_k}{d_{k-1}}\frac{d_{k-1}}{d_{k-2}}\frac{d_{k-2}}{d_{k-3}}\frac{d_{k-3}}{d_{k-4}}\frac{d_{k-4}}{d_{k-5}}\frac{d_{k-5}}{d_{k-6}} + \\
+ \big[Q_4\,\gamma^2\,(k+1)\,k + M_5\,\gamma^2\,(k+1)\big]\;\frac{d_{k+1}}{d_k}\frac{d_k}{d_{k-1}}\frac{d_{k-1}}{d_{k-2}}\frac{d_{k-2}}{d_{k-3}}\frac{d_{k-3}}{d_{k-4}}\frac{d_{k-4}}{d_{k-5}}\frac{d_{k-5}}{d_{k-6}} = 0,
\end{aligned}
$$

and tend $k \to \infty$, so obtaining an algebraic equation for parameter R:

$$
4R^3 + 4\gamma R^4 + \gamma^2 R^5 = 0 \quad \Rightarrow \quad R = 0\,,\; -\frac{2}{\gamma}\,.
$$

so, there are possible convergence radii:

$$R_{\rm conv} = \frac{1}{|R|} = \frac{|\gamma|}{2},\ \infty\,. \tag{73}$$

As a quantization rule we use the transcendence condition. To do this in 8-term recurrence relation

$$\begin{aligned} D_{k-6} d_{k-6} + D_{k-5} d_{k-5} + D_{k-4} d_{k-4} + D_{k-3} d_{k-3} +\\ + D_{k-2} d_{k-2} + D_{k-1} d_{k-1} + D_k d_k + D_{k+1} d_{k+1} = 0\,. \end{aligned} \tag{74}$$

we require vanishing the coefficient $D_{k-6},\ k \geq 6$: $D_{k-6} = 4N_1 + 2n_1 = 0$, where

$$\begin{aligned} N_1 = \frac{1}{\gamma^3}\,(4\,a_5\,B\gamma^2 + 4\,a_6\,B\gamma + 2\,b_4\,\gamma^2 - 4\,a_5\,\gamma - 8\,a_6 + \\ + 2\,b_5\,\gamma + c_1\,\gamma^3 + a_1\,B^2\gamma^3 + b_1\,B\gamma^3 - 24\,B^2\gamma^2 + 48\,B\gamma + 16\,\epsilon\,B\gamma^3 + 18\,B^3\gamma^3)\,, \\ n_1 = \frac{1}{\gamma^3}\,(-8\,a_5\,B\gamma^2 - 8\,a_6\,B\gamma + 8\,a_5\,\gamma + c_7\,\gamma^3 - 4\,b_5\,\gamma - 4\,b_4\,\gamma^2 + \\ + 16\,a_6 + a_5\,B^2\gamma^3 + b_4\,B\gamma^3 + 48\,B^2\gamma^2 - 4\,B^3\gamma^3 - 96\,B\gamma) = 0\,. \end{aligned}$$

Taking into consideration the explicit form of N_1 and n_1, we get

$$D_{k-6} = 64\,B\epsilon + 2\,b_4\,B + 2\,a_5\,B^2 + 4\,b_1\,B + 4\,a_1\,B^2 + 4\,c_1 + 64\,B^3 + 2\,c_7\,.$$

Accounting the expressions for a_1, a_5, b_1, b_4, c_1, c_7, we derive

$$D_{k-6} = 64\,(B + \frac{1}{4}\,\alpha)\,\left(B^2 + 2\,\epsilon\right)\,. \tag{75}$$

Because of $B = \pm\sqrt{-2\epsilon}$ the coefficient D_{k-6} vanishes identically. It means that the structure of the power series is described by 7-term recurrence relations.

Let us study the transcendency condition for this 7-term power series, assuming the coefficient at d_{n-5} vanishes:

$$4Q_0(k-5)(k-6) + (4M_1 + 2m_1)\,(k-5) + (4N_1\gamma + 4\,N_2 + \gamma n_1 + n_2) = 0\,,$$

where

$$\begin{aligned} B = -\sqrt{-2\epsilon}\,, \quad Q_0 = 4\epsilon + 6B^2\,, \\ M_1 = \frac{4\,a_5\,\gamma + 48 + b_1\,\gamma^2 + 16\,\epsilon\,\gamma^2 + 4\,a_6 - 48\,B\gamma + 2\,a_1\,B\gamma^2 + 54\,B^2\gamma^2}{\gamma^2}\,, \\ m_1 = \frac{-96 + b_4\,\gamma^2 - 8\,a_6 - 8\,a_5\,\gamma + 96\,B\gamma + 2\,a_5\,B\gamma^2 - 12\,B^2\gamma^2}{\gamma^2}\,, \\ N_1 = \frac{1}{\gamma^3}\,(4\,a_5\,B\gamma^2 + 4\,a_6\,B\gamma + 2\,b_4\,\gamma^2 - 4\,a_5\,\gamma - 8\,a_6 + \\ + 2\,b_5\,\gamma + c_1\,\gamma^3 + a_1\,B^2\gamma^3 + b_1\,B\gamma^3 - 24\,B^2\gamma^2 + 48\,B\gamma + 16\,\epsilon\,B\gamma^3 + 18\,B^3\gamma^3)\,, \\ N_2 = \frac{8\,\epsilon\,\gamma^2 + 2\,a_5\,\gamma + c_2\,\gamma^2 + 2\,b_1\,\gamma^2 + 2\,a_6 + a_2\,B^2\gamma^2 + b_2\,B\gamma^2 - 24\,B\gamma + 4\,a_1\,B\gamma^2 + 72\,B^2\gamma^2}{\gamma^2}\,, \\ n_1 = \frac{1}{\gamma^3}\,(-8\,a_5\,B\gamma^2 - 8\,a_6\,B\gamma + 8\,a_5\,\gamma + c_7\,\gamma^3 - 4\,b_5\,\gamma - 4\,b_4\,\gamma^2 + \\ + 16\,a_6 + a_5\,B^2\gamma^3 + b_4\,B\gamma^3 + 48\,B^2\gamma^2 - 4\,B^3\gamma^3 - 96\,B\gamma) = 0\,, \\ n_2 = \frac{-8\,a_6\,B\gamma + c_8\,\gamma^2 - 4\,b_5\,\gamma + 8\,a_6 + a_6\,B^2\gamma^2 + b_5\,B\gamma^2}{\gamma^2}\,. \end{aligned}$$

Further we obtain

$$\begin{aligned} 68\,B^3\gamma + \left[(a_5 + 4\,a_1)\,\gamma + a_6 - 72\,k + 24\,k^2 + 4\,a_2\right] B^2 + \\ + \left[(b_4 + 4\,b_1 + 64\,\epsilon)\,\gamma + (8\,a_1 + 4\,a_5)\,k + 4\,b_2 + b_5 - 12\,a_5 - 24\,a_1\right] B + \\ + (4\,c_1 + c_7)\,\gamma + 16\,\epsilon\,k^2 + (2\,b_4 - 112\,\epsilon + 4\,b_1)\,k + c_8 + 192\,\epsilon + 4\,c_2 - 6\,b_4 - 12\,b_1 = 0\,, \end{aligned}$$

which, with explicit expressions for a_1, a_2, a_5, ..., takes the form:

$$68\,B^3\gamma + \left[48 + 16\,\alpha\,\gamma - 72\,k - 16\,\nu^2 + 24\,k^2\right] B^2 + \left[136\,\epsilon\,\gamma + (32\,k - 48)\,\alpha\right] B + \\ + \left(32 - 32\,\nu^2 + 32\,\alpha\,\gamma + 16\,k^2 - 48\,k\right)\epsilon + 16\,\alpha^2 = 0\,.$$

With equality $B = -\sqrt{-2\,\epsilon})$ in mind, we derive a quadratic equation with respect to B, and get the following solutions:

$$\epsilon_1 = -\frac{1}{2}\,\frac{\alpha^2}{(k-2)^2}\,, \quad \epsilon_2 = -\frac{1}{2}\,\frac{\alpha^2}{(k-1)^2}\,. \tag{76}$$

These expressions for energy formulae do not include the quantum number j, so they are unlikely to be relevant to correct energy spectra.

Let us examine solutions of the type III (67). To describe bound states, parameter γ must be negative. When $A = -1$, $B = -\sqrt{-2\epsilon}$, $C = +\gamma$ the equation (61) takes form:

$$\frac{d^4 f}{dx^4} + \left[4\,B + \frac{6}{x} - \frac{4\,\gamma}{x^2} - \frac{4}{2\,x+\gamma}\right]\frac{d^3 f}{dx^3} + \\
+ \left[6\,B^2 + 4\,\epsilon + \frac{-12\,\gamma + a_1\,\gamma^2 + 18\,B\gamma^2}{\gamma^2 x} + \frac{-6\,\gamma + a_2\,\gamma - 12\,B\gamma^2}{\gamma x^2} - \frac{8\,\gamma}{x^3} + \frac{5\,\gamma^2}{x^4} + \right. \\
\left. + \frac{24\,\gamma - 12\,B\gamma^2 + a_5\,\gamma^2}{(2\,x+\gamma)\,\gamma^2} + \frac{a_6}{(2\,x+\gamma)^2}\right]\frac{d^2 f}{dx^2} + \\
+ \left[4\,B^3 + 8\,\epsilon\,B + \frac{-8\,\epsilon\,\gamma^4 + 2\,a_1\,B\gamma^4 - 48\,\gamma^2 + 18\,B^2\gamma^4 + 2\,a_5\,\gamma^3 + 6\,a_6\,\gamma^2 - 24\,B\gamma^3 + b_1\,\gamma^4}{\gamma^4 x} + \right. \\
+ \frac{-8\,\epsilon\,\gamma^4 - 2\,a_1\,\gamma^3 + 2\,a_2\,B\gamma^3 + 24\,\gamma^2 - 12\,B\gamma^3 - 12\,B^2\gamma^4 - 2\,a_5\,\gamma^3 - 2\,a_6\,\gamma^2 + b_2\,\gamma^3}{\gamma^3 x^2} + \\
+ \frac{-2\,a_1\,\gamma^3 - 2\,a_2\,\gamma^2 - 16\,B\gamma^3 + 12\,\gamma^2 + b_3\,\gamma^2}{\gamma^2 x^3} + \frac{-2\,a_2\,\gamma^2 + 40\,\gamma^2 + 10\,B\gamma^3}{\gamma x^4} - \frac{2\,\gamma^3}{x^6} + \\
\left. + \frac{2a_5 B\gamma^4 + 96\gamma^2 - 4a_5\gamma^3 - 12a_6\gamma^2 + 48B\gamma^3 - 12B^2\gamma^4 + b_4\gamma^4}{(2x+\gamma)\,\gamma^4} + \frac{2a_6 B\gamma^2 - 4a_6\gamma + b_5\gamma^2}{(2x+\gamma)^2\,\gamma^2}\right]\frac{df}{dx} + \\
+ \left[\frac{1}{\gamma^6 x}\left(-32\,\gamma^3 + c_1\,\gamma^6 + 6\,B^3\gamma^6 + 2\,a_5\,B\gamma^5 + 6\,a_6\,B\gamma^4 - 48\,B\gamma^4 + 4\,a_5\,\gamma^4 + \right.\right. \\
\left. + 8\,a_6\,\gamma^3 + b_4\,\gamma^5 - 12\,B^2\gamma^5 + 3\,b_5\,\gamma^4 + b_1\,B\gamma^6 + a_1\,B^2\gamma^6 - 8\,\epsilon\,B\gamma^6\right) + \\
+ \frac{1}{\gamma^5 x^2}\left(16\,\gamma^3 - b_1\,\gamma^5 + c_2\,\gamma^5 - 4\,B^3\gamma^6 - 2\,a_5\,B\gamma^5 - 2\,a_6\,B\gamma^4 + 24\,B\gamma^4 - 2\,a_5\,\gamma^4 - 2\,a_6\,\gamma^3 - \right. \\
\left. - b_4\,\gamma^5 - 6\,B^2\gamma^5 - b_5\,\gamma^4 + b_2\,B\gamma^5 + a_2\,B^2\gamma^5 - 8\,\epsilon\,B\gamma^6 + 8\,\epsilon\,\gamma^5 - 2\,a_1\,B\gamma^5\right) + \\
+ \frac{1}{\gamma^4 x^3}\left(-8\,\gamma^3 - b_1\,\gamma^5 - b_2\,\gamma^4 + 2\,a_1\,\gamma^4 + c_3\,\gamma^4 + \right. \\
\left. + 12\,B\gamma^4 + 2\,a_5\,\gamma^4 - 8\,B^2\gamma^5 + b_3\,B\gamma^4 + 16\,\epsilon\,\gamma^5 - 2\,a_1\,B\gamma^5 - 2\,a_2\,B\gamma^4\right) + \\
+ \frac{-20\gamma^3 - b_2\gamma^4 + 4a_1\gamma^4 + 2a_2\gamma^3 - b_3\gamma^3 + c_4\gamma^3 + 40B\gamma^4 + a_5\gamma^4 + a_6\gamma^3 + 5B^2\gamma^5 + 4\epsilon\gamma^5 - 2a_2 B\gamma^4}{\gamma^3 x^4} + \\
+ \frac{-60\,\gamma^3 + a_1\,\gamma^4 + 4\,a_2\,\gamma^3 - b_3\,\gamma^3 + c_5\,\gamma^2}{\gamma^2 x^5} + \frac{a_2\,\gamma^3 - 22\,\gamma^3 - 2\,B\gamma^4}{\gamma x^6} + \\
+ \frac{1}{(2\,x+\gamma)\,\gamma^6}\left(64\,\gamma^3 + a_5\,B^2\gamma^6 + b_4\,B\gamma^6 - 4\,B^3\gamma^6 - 4\,a_5\,B\gamma^5 - 12\,a_6\,B\gamma^4 + c_7\,\gamma^6 + \right. \\
\left. + 96\,B\gamma^4 - 8\,a_5\,\gamma^4 - 16\,a_6\,\gamma^3 - 2\,b_4\,\gamma^5 + 24\,B^2\gamma^5 - 6\,b_5\,\gamma^4\right) + \\
\left. + \frac{a_6\,B^2\gamma^4 + b_5\,B\gamma^4 - 8\,a_6\,\gamma^2 - 2\,b_5\,\gamma^3 + c_8\,\gamma^4 - 4\,a_6\,B\gamma^3}{(2\,x+\gamma)^2\,\gamma^4}\right] f = 0\,. \tag{77}$$

Shortly this equation can be written as

$$\frac{d^4f}{dx^4}+\left[P_0+\frac{P_1}{x}+\frac{P_2}{x^2}+\frac{p_1}{2x+\gamma}\right]\frac{d^3f}{dx^3}+\left[Q_0+\frac{Q_1}{x}+\frac{Q_2}{x^2}+\frac{Q_3}{x^3}+\frac{Q_4}{x^4}+\frac{q_1}{2x+\gamma}+\frac{q_2}{(2x+\gamma)^2}\right]\frac{d^2f}{dx^2}+$$
$$+\left[M_0+\frac{M_1}{x}+\frac{M_2}{x^2}+\frac{M_3}{x^3}+\frac{M_4}{x^4}+\frac{M_6}{x^6}+\frac{m_1}{2x+\gamma}+\frac{m_2}{(2x+\gamma)^2}\right]\frac{df}{dx}+$$
$$+\left[\frac{N_1}{x}+\frac{N_2}{x^2}+\frac{N_3}{x^3}+\frac{N_4}{x^4}+\frac{N_5}{x^5}+\frac{N_6}{x^6}+\frac{n_1}{2x+\gamma}+\frac{n_2}{(2x+\gamma)^2}\right]f=0\,. \tag{78}$$

Then the solutions are constructed as power series: $\Phi=\sum_{l=0}^{\infty}d_l x^l$, we derive 9-term recurrence relations ($k\geq 7$:

$$[4\,M_0\,(k-7)+(4\,N_1+2\,n_1)]\;d_{k-7}+$$
$$+\,[4\,Q_0\,(k-6)(k-7)+(4\,M_0\,\gamma+4\,M_1+2\,m_1)\;(k-6)+(4\,N_1\,\gamma+4\,N_2+\gamma n_1+n_2)]\;d_{k-6}+$$
$$+[4\,P_0\,(k-5)(k-6)(k-7)+(4\,Q_0\,\gamma+4\,Q_1+2\,q_1)\;(k-5)(k-6)+$$
$$+\left(M_0\,\gamma^2+4\,M_1\,\gamma+4\,M_2+\gamma m_1+m_2\right)\;(k-5)+\left(N_1\,\gamma^2+4\,N_2\,\gamma+4\,N_3\right)]\,d_{k-5}+$$
$$+[4\,(k-4)(k-5)(k-6)(k-7)+(4\,P_0\,\gamma+4\,P_1+2\,p_1)\;(k-4)(k-5)(k-6)+$$
$$+\left(P_0\gamma^2+4P_1\gamma+4P_2+\gamma p_1\right)(k-4)(k-5)(k-6)+\left(Q_0\gamma^2+4Q_1\gamma+4Q_2+q_1\gamma+q_2\right)(k-4)(k-5)+$$
$$+\left(M_1\,\gamma^2+4\,M_2\,\gamma+4\,M_3\right)\;(k-4)+\left(N_2\,\gamma^2+4\,N_3\,\gamma+4\,N_4\right)]\,d_{k-4}+$$
$$+[4\,\gamma\,(k-3)(k-4)(k-5)(k-6)+\left(Q_1\,\gamma^2+4\,Q_2\,\gamma+4\,Q_3\right)\;(k-3)(k-4)+$$
$$+\left(M_2\,\gamma^2+4\,M_3\,\gamma+4\,M_4\right)\;(k-3)+\left(N_3\,\gamma^2+4\,N_4\,\gamma+4\,N_5\right)]\,d_{k-3}+$$
$$+[\gamma^2\,(k-2)(k-3)(k-4)(k-5)+\left(P_1\,\gamma^2+4\,P_2\,\gamma\right)\;(k-2)(k-3)(k-4)+$$
$$+\left(Q_2\gamma^2+4Q_3\gamma+4Q_4\right)(k-2)(k-3)+\left(M_3\gamma^2+4M_4\gamma\right)(k-2)+\left(N_4\gamma^2+4N_5\gamma+4N_6\right)]d_{k-2}+$$
$$+[P_2\,\gamma^2\,(k-1)(k-2)(k-3)+\left(Q_3\,\gamma^2+4\,Q_4\,\gamma\right)\;(k-1)(k-2)+$$

$$+\left(M_4\,\gamma^2+4\,M_6\right)\;(k-1)+\left(N_5\,\gamma^2+4\,N_6\,\gamma\right)]\,d_{k-1}+$$
$$+[Q_4\,\gamma^2\,k(k-1)+4\,M_6\,\gamma\,k+N_6\,\gamma^2]\,d_k+M_6\,\gamma^2\,(k+1)d_{k+1}=0\,. \tag{79}$$

To study the convergence of the power series, we divide the expression by $k^4 d_{k-7}$ and tend $k\to\infty$. As a result we obtain ane algebraic equation for the quantity $R=\lim_{k\to\infty}\frac{d_{k+1}}{d_k}$:

$$4R^3+4\gamma R^4+\gamma^2R^5=0\quad\Rightarrow R=0,\;-\frac{2}{\gamma}\,,\quad R_{conv}=\frac{1}{|R|}=\frac{|\gamma|}{2},\infty\,. \tag{80}$$

The quantization rule is searched from transcendency condition. So we demand vanishing the coefficient D_{k-7}:

$$4\,M_0\,(k-7)+4\,N_1+2\,n_1=0\,.$$

Taking into account the explicit form of the parameters $M_0\,,\;N_1\,,\;n_1$

$$(16\,k-96)\,B^3+(2\,a_5+4\,a_1)\,B^2+[(32\,k-256)\,\epsilon+4\,b_1+2\,b_4]\,B+4\,c_1+2\,c_7=0\,,$$

we get the identity $0=0$. So, we have 8-term recurrence relations in (79). Therefore, to find a quantization rule, the coefficient at d_{k-6} has to be zero:

$$4\,Q_0\,(k-6)(k-7)+(4\,M_0\,\gamma+4\,M_1+2\,m_1)\;(k-6)+4\,N_1\,\gamma+4\,N_2+\gamma n_1+n_2=0$$

or

$$(16\,\gamma k-92\,\gamma)\,B^3+\left(720+16\,\alpha\,\gamma+24\,k^2-264\,k-16\,\nu^2\right)B^2+$$
$$+\left((32\,\gamma k-184\,\gamma)\,\epsilon-176\,\alpha+32\,\alpha\,k\right)B+\left(480+32\,\alpha\,\gamma-32\,\nu^2-176\,k+16\,k^2\right)\epsilon+16\,\alpha^2=0\,.$$

As $B=-\sqrt{-2\epsilon}$, we get the following equation with respect to ϵ:

$$-32\;(k-5)\,(k-6)\;\epsilon+(176-32\,k)\,\alpha\,\sqrt{-2\epsilon}+16\,\alpha^2=0\,,$$

whence we derive

$$\epsilon_1 = -\frac{1}{2}\,\frac{\alpha^2}{(k-5)^2}\,, \quad \epsilon_2 = -\frac{1}{2}\,\frac{\alpha^2}{(k-6)^2}\,. \tag{81}$$

These spectra are unlikely to be correct as well.

Now, we consider the solutions of 4-th type(67): IV, $\Psi(x) = e^{Bx} x e^{-\gamma/x} f_4(x)$. At chosen values of the parameters the equation (61) takes a form

$$\begin{aligned}
&\frac{d^4 f}{dx^4} + \left[4\,B + \frac{14}{x} + \frac{4\,\gamma}{x^2} - \frac{4}{2\,x+\gamma}\right]\frac{d^3 f}{dx^3} + \\
&+\left[6\,B^2 + 4\,\epsilon + \frac{12\,\gamma + a_1\,\gamma^2 + 42\,B\gamma^2}{\gamma^2 x} + \frac{18\,\gamma + a_2\,\gamma + 12\,B\gamma^2}{\gamma x^2} + \frac{28\,\gamma}{x^3} + \frac{5\,\gamma^2}{x^4} + \right.\\
&\left. + \frac{-24\,\gamma - 12\,B\gamma^2 + a_5\,\gamma^2}{(2\,x+\gamma)\,\gamma^2} + \frac{a_6}{(2\,x+\gamma)^2}\right]\frac{d^2 f}{dx^2} + \\
&+\left[4\,B^3 + 8\,\epsilon\,B + \frac{8\,\epsilon\,\gamma^4 + 2\,a_1\,B\gamma^4 + 96\,\gamma^2 + 42\,B^2\gamma^4 - 2\,a_5\,\gamma^3 - 6\,a_6\,\gamma^2 + 24\,B\gamma^3 + b_1\,\gamma^4}{\gamma^4 x} + \right.\\
&+\frac{8\,\epsilon\,\gamma^4 + 2\,a_1\,\gamma^3 + 2\,a_2\,B\gamma^3 - 48\,\gamma^2 + 36\,B\gamma^3 + 12\,B^2\gamma^4 + 2\,a_5\,\gamma^3 + 2\,a_6\,\gamma^2 + b_2\,\gamma^3}{\gamma^3 x^2} + \\
&+\frac{2\,a_1\,\gamma^3 + 2\,a_2\,\gamma^2 + 56\,B\gamma^3 + 24\,\gamma^2 + b_3\,\gamma^2}{\gamma^2 x^3} + \frac{2\,a_2\,\gamma^2 - 16\,\gamma^2 + 10\,B\gamma^3}{\gamma x^4} + \frac{12\,\gamma^2}{x^5} + \frac{2\,\gamma^3}{x^6} + \\
&\left.+\frac{2a_5 B\gamma^4 - 192\gamma^2 + 4a_5\gamma^3 + 12a_6\gamma^2 - 48B\gamma^3 - 12B^2\gamma^4 + b_4\gamma^4}{(2x+\gamma)\,\gamma^4} + \frac{2a_6 B\gamma^2 + 4a_6\gamma + b_5\gamma^2}{(2x+\gamma)^2\,\gamma^2}\right]\frac{df}{dx} + \\
&+\left[\frac{1}{\gamma^6 x}\left(320\,\gamma^3 - 8\,a_5\,\gamma^4 + 96\,B\gamma^4 - 32\,a_6\,\gamma^3 + 12\,B^2\gamma^5 + b_1\,B\gamma^6 + a_1\,B^2\gamma^6 + c_1\,\gamma^6 + \right.\right.\\
&\left. + 14\,B^3\gamma^6 - b_4\,\gamma^5 - 3\,b_5\,\gamma^4 - 2\,a_5\,B\gamma^5 - 6\,a_6\,B\gamma^4 + 8\,\epsilon\,B\gamma^6\right) + \\
&+\frac{1}{\gamma^5 x^2}\left(-160\,\gamma^3 + 2\,a_1\,B\gamma^5 + 4\,a_5\,\gamma^4 - 48\,B\gamma^4 + 12\,a_6\,\gamma^3 + 18\,B^2\gamma^5 + b_2\,B\gamma^5 + a_2\,B^2\gamma^5 + \right.\\
&\left. + c_2\,\gamma^5 + 4\,B^3\gamma^6 + b_4\,\gamma^5 + b_5\,\gamma^4 + 2\,a_5\,B\gamma^5 + 2\,a_6\,B\gamma^4 + b_1\,\gamma^5 + 8\,\epsilon\,B\gamma^6\right) + \\
&+\frac{80\gamma^3 + 2a_1 B\gamma^5 + 2a_2 B\gamma^4 - 2a_5\gamma^4 + 24B\gamma^4 - 4a_6\gamma^3 + 28B^2\gamma^5 + b_3 B\gamma^4 + c_3\gamma^4 + b_1\gamma^5 + b_2\gamma^4}{\gamma^4 x^3} + \\
&+\frac{-40\,\gamma^3 + 2\,a_2\,B\gamma^4 + a_5\,\gamma^4 - 16\,B\gamma^4 + a_6\,\gamma^3 + 5\,B^2\gamma^5 + c_4\,\gamma^3 + 4\,\epsilon\,\gamma^5 + b_2\,\gamma^4 + b_3\,\gamma^3}{\gamma^3 x^4} + \\
&+\frac{20\,\gamma^3 + a_1\,\gamma^4 + b_3\,\gamma^3 + 12\,B\gamma^4 + c_5\,\gamma^2}{\gamma^2 x^5} + \frac{a_2\,\gamma^3 - 20\,\gamma^3 + 2\,B\gamma^4}{\gamma x^6} + \\
&+\frac{1}{(2\,x+\gamma)\,\gamma^6}\left(-640\,\gamma^3 + a_5\,B^2\gamma^6 + 16\,a_5\,\gamma^4 - 192\,B\gamma^4 + 64\,a_6\,\gamma^3 - 24\,B^2\gamma^5 + \right.\\
&\left. + b_4\,B\gamma^6 - 4\,B^3\gamma^6 + c_7\,\gamma^6 + 2\,b_4\,\gamma^5 + 6\,b_5\,\gamma^4 + 4\,a_5\,B\gamma^5 + 12\,a_6\,B\gamma^4\right) + \\
&\left. + \frac{a_6\,B^2\gamma^4 + b_5\,B\gamma^4 + 16\,a_6\,\gamma^2 + 2\,b_5\,\gamma^3 + c_8\,\gamma^4 + 4\,a_6\,B\gamma^3}{(2\,x+\gamma)^2\,\gamma^4}\right] f = 0\,.
\end{aligned} \tag{82}$$

Schematically the equation can be written as follows:

$$\begin{aligned}
&\frac{d^4 f}{dx^4} + \left[P_0 + \frac{P_1}{x} + \frac{P_2}{x^2} + \frac{p_1}{2x+\gamma}\right]\frac{d^3 f}{dx^3} + \\
&+\left[Q_0 + \frac{Q_1}{x} + \frac{Q_2}{x^2} + \frac{Q_3}{x^3} + \frac{Q_4}{x^4} + \frac{q_1}{2x+\gamma} + \frac{q_2}{(2x+\gamma)^2}\right]\frac{d^2 f}{dx^2} + \\
&+\left[M_0 + \frac{M_1}{x} + \frac{M_2}{x^2} + \frac{M_3}{x^3} + \frac{M_4}{x^4} + \frac{M_5}{x^5} + \frac{M_6}{x^6} + \frac{m_1}{2x+\gamma} + \frac{m_2}{(2x+\gamma)^2}\right]\frac{df}{dx} + \\
&+\left[\frac{N_1}{x} + \frac{N_2}{x^2} + \frac{N_3}{x^3} + \frac{N_4}{x^4} + \frac{N_5}{x^5} + \frac{N_6}{x^6} + \frac{n_1}{2x+\gamma} + \frac{n_2}{(2x+\gamma)^2}\right] f = 0\,.
\end{aligned} \tag{83}$$

Its solutions are constructed as power series; we derive 9-term recurrence relations:

$\underline{k \geq 7,}$

$$
\begin{aligned}
&[4\, M_0\, (k-7) + (4\, N_1 + 2\, n_1)]\, d_{k-7} + \\
&+[4\, Q_0\, (k-6)(k-7) + (4\, M_0\, \gamma + 4\, M_1 + 2\, m_1)\ (k-6) + (4\, N_1\, \gamma + 4\, N_2 + \gamma n_1 + n_2)]\, d_{k-6} + \\
&+[4\, P_0\, (k-5)(k-6)(k-7) + (4\, Q_0\, \gamma + 4\, Q_1 + 2\, q_1)\ (k-5)(k-6) + \\
&+ \left(M_0\, \gamma^2 + 4\, M_1\, \gamma + 4\, M_2 + \gamma m_1 + m_2\right)\ (k-5) + \left(N_1\, \gamma^2 + 4\, N_2\, \gamma + 4\, N_3\right)]\, d_{k-5} + \\
&+[4\, (k-4)(k-5)(k-6)(k-7) + (4\, P_0\, \gamma + 4\, P_1 + 2\, p_1)\ (k-4)(k-5)(k-6) + \\
&+ \left(P_0 \gamma^2 + 4 P_1 \gamma + 4\, P_2 + \gamma p_1\right) (k-4)(k-5)(k-6) + \left(Q_0 \gamma^2 + 4 Q_1 \gamma + 4 Q_2 + q_1 \gamma + q_2\right) (k-4)(k-5) + \\
&+ \left(M_1\, \gamma^2 + 4\, M_2\, \gamma + 4\, M_3\right)\ (k-4) + \left(N_2\, \gamma^2 + 4\, N_3\, \gamma + 4\, N_4\right)]\, d_{k-4} + \\
&+[4\, \gamma\, (k-3)(k-4)(k-5)(k-6) + \left(Q_1\, \gamma^2 + 4\, Q_2\, \gamma + 4\, Q_3\right)\ (k-3)(k-4) + \\
&+ \left(M_2\, \gamma^2 + 4\, M_3\, \gamma + 4\, M_4\right)\ (k-3) + \left(N_3\, \gamma^2 + 4\, N_4\, \gamma + 4\, N_5\right)]\, d_{k-3} + \\
&+[\gamma^2\, (k-2)(k-3)(k-4)(k-5) + \left(P_1\, \gamma^2 + 4\, P_2\, \gamma\right)\ (k-2)(k-3)(k-4) + \\
&+ \left(Q_2 \gamma^2 + 4 Q_3 \gamma + 4 Q_4\right) (k-2)(k-3) + \left(M_3 \gamma^2 + 4 M_4 \gamma + 4 M_5\right) (k-2) + \left(N_4 \gamma^2 + 4 N_5 \gamma + 4 N_6\right)] d_{k-2} + \\
&+[P_2\, \gamma^2\, (k-1)(k-2)(k-3) + \left(Q_3\, \gamma^2 + 4\, Q_4\, \gamma\right)\ (k-1)(k-2) + \\
&+ \left(M_4\, \gamma^2 + 4\, M_5\, \gamma + 4\, M_6\right)\ (k-1) + \left(N_5\, \gamma^2 + 4\, N_6\, \gamma\right)]\, d_{k-1} + \\
&+[Q_4\, \gamma^2\, k(k-1) + \left(M_5\, \gamma^2 + 4\, M_6\, \gamma\right)\ k + N_6\, \gamma^2]\, d_k + M_6\, \gamma^2\, (k+1) d_{k+1} = 0\, .
\end{aligned} \tag{84}
$$

To study the convergence of the power series, we divide the expression by $k^4 d_{k-7}$ and tending g $k \to \infty$, we get the algebraic equation for convergence radius:

$$
R = \lim_{k \to \infty} \frac{d_{k+1}}{d_k}, \quad 4R^3 + 4\gamma R^4 + \gamma^2 R^5 = 0 \quad \Rightarrow R = 0,\ -\frac{2}{\gamma}\, , \quad R_{conv} = \frac{1}{|R|} = \frac{|\gamma|}{2},\ \infty\, . \tag{85}
$$

Considering the coefficient at D_{k-7}, $k \geq 8$:

$$
4\, M_0\, (k-7) + 4\, N_1 + 2\, n_1 = 0\, ,
$$

and taking into account the explicit expressions of parameters $M_0\, ,\ N_1\, ,\ n_1$

$$
(16\, k - 64)\, B^3 + (2\, a_5 + 4\, a_1)\, B^2 + [(32\, k - 192)\, \epsilon + 4\, b_1 + 2\, b_4]\, B + 4\, c_1 + 2\, c_7 = 0\, ,
$$

we get the identity $0 = 0$. So, there is 8-term recurrence relations. The transcendence condition means that the coefficient at d_{k-6} vanishes:

$$
4\, Q_0\, (k-6)(k-7) + (4\, M_0\, \gamma + 4\, M_1 + 2\, m_1)\ (k-6) + (4\, N_1\, \gamma + 4\, N_2 + \gamma n_1 + n_2)
$$

or

$$
\left(-32\, k^2 - 384 + 224\, k\right) \epsilon - 32\, \left(k - \frac{7}{2}\right) \alpha\, \sqrt{-2\epsilon} + 16\, \alpha^2 = 0\, ,
$$

then, the quantization rule is follows:

$$
\epsilon_1 = -\frac{1}{2}\, \frac{\alpha^2}{(k-3)^2}\, , \quad \epsilon_2 = -\frac{1}{2}\, \frac{\alpha^2}{(k-4)^2}\, . \tag{86}
$$

These spectra also are unlikely to be correct.

All the constructed solutions are exact, but they are formal because any reliable rules for quantization of energy levels are not known. The transcendency condition is solving this difficulty only partly.

VII. KCC-GEOMETRICAL APPROACH

In this section we will try to study arising in the above mathematical tasks by applying special geometric method based on so called KCC-invariants [15] – [18]. In this approach, one considers a system of second order differential equations

$$
\dot{y}^{\,i}(r) + 2Q^i(r, x, y) = 0\, , \tag{87}
$$

which corresponds to the the Euler-Lagrange equations for some dynamical system with Lagrangian L. In (87), the symbol x^i designates so called coordinates, their derivatives in argument r are $y^i = dx^i/dr = \dot{x}^i$, and the quantities Q_i are determined through some Lagrangian L as follows

$$Q^i = \frac{1}{4} g^{il} \left(\frac{\partial^2 L}{\partial x^k \partial y^l} y^k - \frac{\partial L}{\partial x^l} + \frac{\partial^2 L}{\partial y^l \partial r} \right), \qquad g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}. \tag{88}$$

The first and second invariants, $\varepsilon^i(r, x, y)$ and P^i_j are introduced by the definitions

$$\varepsilon^i = \frac{\partial Q^i}{\partial y^j} y^j - 2Q^i, \qquad P^i_j = 2\frac{\partial Q^i}{\partial x^j} + 2Q^s \frac{\partial^2 Q^i}{\partial y^j \partial y^s} - \frac{\partial^2 Q^i}{\partial y^j \partial x^s} y^s - \frac{\partial Q^i}{\partial y^s} \frac{\partial Q^s}{\partial y^j} - \frac{\partial^2 Q^i}{\partial y^j \partial r}. \tag{89}$$

The second invariant P^i_j relates to Jacobi stability of dynamical system. There is an analogy between the equations of geodesic deviation expressed in terms of the Riemann curvature and and in terms of the second KCC-invariant:

$$\frac{D^2 \xi^i}{Ds^2} = R^i_{kjl} \frac{dx^k}{ds} \frac{dx^l}{ds} \xi^j = -K^i_j \xi^j, \qquad \frac{D^2 \xi^i}{Dr^2} = P^i_j \xi^j. \tag{90}$$

It is known that a pencil of geodesic curves from the some point r_0 converges (or diverges) if the real parts of all eigenvalues of the invariant $P^i{}_j$ are negative (or positive) ones.

We start from the system of two second-order differential equations for two radial functions of spin 1 particle with electric quadrupole moment in the external Coulomb field, the equation (56). Let us rewrite it as:

$$\begin{gathered}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2m\frac{\alpha + Er}{r} - \frac{2\nu^2}{r^2} - \frac{2\Gamma}{r^3} - \frac{\Gamma^2}{r^4} \right) \Psi_1(r) - \nu \frac{2r + \Gamma}{r^3} \Psi_2(r) = 0, \\ \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2m\frac{\alpha + Er}{r} - \frac{2\nu^2}{r^2} - \frac{2}{r^2} \right) \Psi_2(r) - 2\nu \frac{2r + \Gamma}{r^3} \Psi_1(r) = 0.\end{gathered} \tag{91}$$

We will apply the following notations $x^i = \Psi_i(r)$, $y^i = (d/dr)\Psi_i(r) = \dot{\Psi}_i(r)$. Then comparing equations (91) and (87), one finds the quantities Q^i:

$$\begin{gathered}Q^1(r, \Psi_i, \dot{\Psi}_i) = \left(Em + \frac{\alpha m}{r} - \frac{\Gamma^2}{2r^4} - \frac{\Gamma}{r^3} - \frac{\nu^2}{r^2} \right) \Psi_1 - \nu \frac{(\Gamma + 2r)}{2r^3} \Psi_2 + \frac{1}{r} \dot{\Psi}_1, \\ Q^2(r, \Psi_i, \dot{\Psi}_i) = \left(Em + \frac{\alpha m}{r} - \frac{\nu^2}{r^2} - \frac{1}{r^2} \right) \Psi_2 - \nu \frac{(\Gamma + 2r)}{r^3} \Psi_1 + \frac{1}{r} \dot{\Psi}_2.\end{gathered} \tag{92}$$

By the direct calculation according the formula (89), the first and second KCC-invariants equal

$$\begin{gathered}\varepsilon^1 = \Psi_1 \left(-2Em - \frac{2\alpha m}{r} + \frac{\Gamma^2}{r^4} + \frac{2\Gamma}{r^3} + \frac{2\nu^2}{r^2} \right) + \nu \Psi_2 \left(\frac{\Gamma}{r^3} + \frac{2}{r^2} \right) - \frac{\dot{\Psi}_1}{r}, \\ \varepsilon^2 = 2\Psi_2 \left(-Em - \frac{\alpha m}{r} + \frac{\nu^2 + 1}{r^2} \right) + 2\nu \Psi_1 \left(\frac{\Gamma}{r^3} + \frac{2}{r^2} \right) - \frac{\dot{\Psi}_2}{r};\end{gathered} \tag{93}$$

$$P^i_j = \begin{vmatrix} -\frac{\Gamma^2}{r^4} - \frac{2\Gamma}{r^3} - \frac{2\nu^2}{r^2} + 2Em + \frac{2m\alpha}{r} & -\frac{\nu(2r+\Gamma)}{r^3} \\ -\frac{2\nu(2r+\Gamma)}{r^3} & 2Em + \frac{2\alpha m}{r} - \frac{2(\nu^2+1)}{r^2} \end{vmatrix}. \tag{94}$$

The eigenvalues Λ_1, Λ_2 of the second invariant are given by the formulas

$$\Lambda_{1,2} = 2Em + \frac{2\alpha m}{r} - \frac{\Gamma^2}{2r^4} - \frac{\Gamma}{r^3} - \frac{2\nu^2 + 1}{r^2} \pm \frac{\sqrt{(\Gamma^2 - 2r^2 + 2\Gamma r)^2 + 8\nu^2 r^2 (\Gamma + 2r)^2}}{2r^4}. \tag{95}$$

Typical behavior of eigenvalues at different j is presented in figure 1.

Let study the behavior of the eigenvalues Λ^i near the singular points $r = 0$, $r = \infty$, $r = -\Gamma/2$. It was found out that

$$r \to 0, \ \Lambda^1 \to -\frac{2}{r^2} < 0, \ \Lambda^2 \to -\frac{\Gamma^2}{r^4} < 0; \qquad r \to \infty, \ \Lambda^1, \Lambda^2 \to 2Em < 0; \tag{96}$$

$$r \to -\frac{\Gamma}{2}, \ \Lambda^1 \to 2Em - \frac{4m\alpha}{\Gamma} - \frac{8\nu^2}{\Gamma^2} - \frac{8}{\Gamma^2} < 0, \ \ \Lambda^2 \to 2Em - \frac{4m\alpha}{\Gamma} - \frac{8\nu^2}{\Gamma^2} < 0. \tag{97}$$

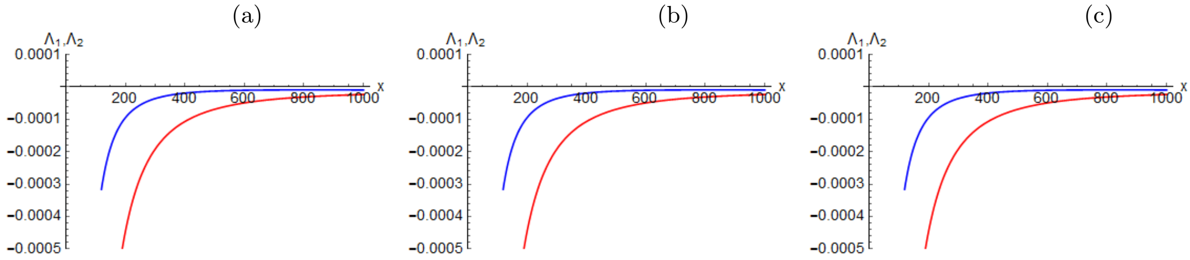

FIG. 1: The dependencies of eigenvalues Λ_1 (red) and Λ_1 (blue) on radial coordinate ($x = mr$) at different j: (a) $j = 1$, (b) $j = 2$, (c) $j = 3$. We used following parameters: $\Gamma m = 1$, $E/m = -0.000009$.

Since the real parts of all eigenvalues of the 2-nd KCC-invariant are negative, the different branches of the solution converges near the singular points $r = 0, \infty, -\Gamma/2$. This correlates with behavior of solutions near the singular points for bound quantum mechanical states (discrete spectra).

The third KCC-invariant

$$R^i_{jk} = \frac{1}{3}\left(\frac{\partial P^i_j}{\partial y^k} - \frac{\partial P^i_k}{\partial y^j}\right) \tag{98}$$

determines the torsion of the Berwald connection.

The fourth KCC-invariant is an extension of the Riemann–Christoffel tensor

$$B^i_{jkl} = \frac{\partial R^i_{jk}}{\partial y^l}. \tag{99}$$

Finally, the fifth KCC-invariant extends the Duglas tensor.

$$D^i_{jkl} = \frac{\partial^3 Q^i}{\partial y^j \partial y^k \partial y^l}. \tag{100}$$

As the vector field Q^i (92) is linear in the coordinates $x^i \equiv \Phi^i$ and $y^i \equiv \dot{\Psi}^i$, the first (93) and second invariants (94) are functions of the radial coordinate r and do not depend on x^i and y^i, and the third, fourth and fifth invariants identically vanish.

The next step is to construct a Lagrangian function L for the phase space $\dot{\Psi}_i$, Ψ_i, such that the formulas for coefficients Q^i (92) hold true, and the dynamics of the system is defined by the equations (91). We will search for the function L in the form

$$L = g_{ij}(r) y^i y^j + b_j(r, x) y^j\,. \tag{101}$$

Let us assume that the tensor g_{ij} is diagonal, $g_{12} = g_{21} = 0$. In this case, substituting (101) into (88) we derive

$$Q^1 = \frac{1}{4g_{11}}\left(2\dot{g}_{11} y^1 + \frac{\partial b_1}{\partial r} + \left(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\right) y^2\right), Q^2 = \frac{1}{4g_{22}}\left(2\dot{g}_{22} y^2 + \frac{\partial b_2}{\partial r} + \left(\frac{\partial b_2}{\partial x^1} - \frac{\partial b_1}{\partial x^2}\right) y^2\right). \tag{102}$$

Equating the terms from (92) with the corresponding terms from (102), we obtain the system of equations with respect to $g_{ij}(r)$ and $b_j(r, x)$:

$$\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1} = 0, \qquad \frac{\dot{g}_{11}}{2g_{11}} = \frac{1}{r}, \qquad \frac{\dot{g}_{22}}{2g_{22}} = \frac{1}{r},$$
$$\frac{1}{4g_1}\frac{\partial b_1}{\partial r} = -\frac{x^1\left(\Gamma^2 - 2r^2\left(mr(\alpha + Er) - \nu^2\right) + 2\Gamma r\right)}{2r^4} - \frac{\nu x^1(\Gamma + 2r)}{2r^3},$$
$$\frac{1}{4g_2}\frac{\partial b_2}{\partial r} = -\frac{x^2\left(-Emr^2 + \nu^2 - \alpha m r + 1\right)}{r^2} - \frac{\nu x^1(\Gamma + 2r)}{r^3}.$$

Its solution is given by the formulas

$$g_{11} = 2C_1 r^2, \quad g_{22} = C_1 r^2,$$
$$b_1 = B_1(x^1, x^2) - 2C_1 \left\{ -\frac{2}{3} Emr^3 x^1 - \alpha m r^2 x^1 - \frac{\Gamma^2 x^1}{r} + \ln r (\Gamma \nu x^2 + 2\Gamma x^1) + 2\nu r(\nu x^1 + x^2) \right\},$$
$$b_2 = B_2(x^1, x^2) - 4C_1 \left\{ -\frac{1}{3} Emr^3 x^2 - \frac{1}{2} \alpha m r^2 x^2 + \Gamma \nu x^1 \ln r + r\left(2\nu x^1 + \nu^2 x^2 + x^2\right) \right\},$$

where C_1 are the arbitrary constant. Two functions $B_1(x^1, x^2)$ and $B_2(x^1, x^2)$ obey the following restriction

$$\frac{\partial B_1(x^1, x^2)}{\partial x^2} - \frac{\partial B_2(x^1, x^2)}{\partial x^1} = 0. \tag{103}$$

In accordance with the known theorem, from (103) we conclude that this 2-dimensional vector field B_1, B_2 can be presented as a gradient of a scalar function

$$B_1(x^1, x^2) = \frac{\partial}{\partial x^1} \varphi(x^1, x^2), \quad B_2(x^1, x^2) = \frac{\partial}{\partial x^2} \varphi(x^1, x^2), \quad B_i = \text{grad}\, \varphi\,. \tag{104}$$

There exist some freedom in choosing the Lagrangian (the constant C_1 may be taken as 1) :

$$L = 2r^2 (y^1)^2 + r^2 (y^2)^2 + 4x^1 y^1 \left(\frac{2}{3} Emr^3 + \alpha m r^2 + \frac{\Gamma^2}{r} - 2\Gamma \ln r - 2\nu^2 r \right) +$$
$$+ \frac{2}{3} r x^2 y^2 \left(mr(3\alpha + 2Er) - 6\left(\nu^2 + 1\right) \right) - 4\nu\left(x^2 y^1 + x^1 y^2\right)(\Gamma \ln r + 2r) +$$
$$+ y^1 \frac{\partial \varphi}{\partial x^1} + y^2 \frac{\partial \varphi}{\partial x^2}, \quad \varphi = \varphi(x^1, x^2). \tag{105}$$

VIII. CONCLUSION

In present work the quantum-mechanical problem of spin 1 particle with additional quadrupole moment in the external Coulomb field has been studied. The variables separation in generalized Duffin-Kemmer equation has been performed through diagonalization of the energy operator, operators of the square and the third projection of total momentum. The equation system for ten radial functions has been derived. By diagonalization of spatial reflection operator the equation system is separated into two subsystems from four and six equations for parities $P = (-1)^{j+1}$ $P = (-1)^j$, respectively. Additional terms provided by the electric quadrupole moment are present in both subsystems.

The system of four relativistic radial equations leads to second-order differential equation for the main function. This equation has two irregular singular points of rank 3 and 2, and four regular points with simple indexes. The Frobenius solutions has been constructed as a power series, 8-term recurrence relations has been found and the power convergence has been studied. The transcendence condition for the solutions gives the formula for energy levels that is suitable from physical point of view.

The relativistic system of six equations turns out to be very complicated. To simplify it the nonrelativistic approximation has been performed. In this case, the radial system is reduced to two linked differential equations of the second order for two functions. Utilizing exclusion method we get two single-type equations of the forth order for these functions. The Frobenius solutions of the equations has been constructed, and the convergence of corresponding 8- and 9-term power series has been studied. Among all solutions we picked out the solutions which could describe the bound states of the particle.

All found Frobenius solutions of the 2-d order and 4-th order equations are exact, but the energy quantization rules are unknown.

We have used a geometrical KCC-based method to study the problem of spin 1 particle with anomalous magnetic moment in the external Coulomb field. The first and the second invariants were calculated. It has been shown that the different branches of the solution converges near the singular points $\infty, -\Gamma/2$, and may converge either diverge near the singular points $r = 0$. This correlates with behavior of solutions near these points for quantum mechanical bound states. The Lagrangians corresponding to the geometrical problem has been found, they are demonstrated to have the arbitrariness up to some special term, which may be considered as specific gauge freedom.

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