On The Binomial Distribution Derivation For The Reactor Core Nuclei Radioactive Decay Description Within Fock's Quasi-Stationary States Theorem

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Classical experimental nuclear physics determines radioactive decay for a close to infinite number of nuclei as process described with a continuous differential function. Such an assumption is valid until one meets a finite amount of a radioactive nuclei in a system (e.g. ultra heavy elements). In that case a binomial distribution must be applied instead of Poisson as well as the intensity of radioactive decay must depend on number of particles in a system. A new approach for describing nuclei radioactive decay was derived in current work via Fock's quasi-stationary states concept. The Poisson and binomial distributions notions were applied for this purpose. Nuclei radioactive decay kinetics equations and asymptotic function form for an average nuclei number in a system has been derived by the means of probability analysis.

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1. INTRODUCTION

In earlier works [1, 2] the birth and death model mathematical apparatus has been studied for describing processes of neutron interaction within reactor core breeding medium. Such model is based on a general probabilistic theory and in particular case on the Kolmogorov differential equations [3]. Further works on this topic [4, 5] were aimed at describing some mathematical issues of the model in general. Hence, on the basis of this general description a more specific problem namely the nuclei radioactive decay must be examined in a more detailed approximation within the model framework and its variations.

In [6] the general radioactive decay equation is defined as

$$-dn = \lambda n dt, \tag{1}$$

which is fair only for the case with infinite nuclei number and when $n(t)$ function can be considered as a continuous one. This condition does not take place for a systems with small number of radioactive nuclei and one has to implement the binomial distribution.

Considering Fock's reference [7] to the quasi-stationary states decay law the binomial distribution seems to be a suitable for describing radioactive decay process. The provisions in [8] also indicate the validity of such a statement. Quasi-stationary states include states in which the uncertainty of the system energy is small compared with its average energy. According

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to the Fock's formulation, the character of state decay over time is determined by the energy distribution function in this state.The theory of this phenomenon is developed in the so-called "Fock-Krylov" theorem.

Current work aims at describing the nuclei radioactive decay on the basis of Fock's quasi-stationary theorem statements. Both Poisson and binomial distributions are assumed to provide a mathematical description of this process.

2. QUASI-STATIONARY THEOREM PROVISIONS

Let $\psi_0 = \psi(x, 0)$ be the initial value of the wave function $\psi(x, t)$ of the system under consideration and x be the set of variables through which the wave function is expressed.The function ψ_0 decomposes into an integral with respect to the eigenfunction $\psi_E(x)$ of the energy operator

$$\psi(x, 0) = \int C(E)\psi_E(x)dE. \tag{2}$$

The state of the system in time t will be determined then by the integral

$$\psi(x, 0) = \int e^{-\frac{iEt}{\hbar}} C(E)\psi_E(x)dE, \tag{3}$$

and the probability $L(t)$ of the fact that after time t the system can be detected in the initial state is equal to the square of the module of the scalar product:

$$L(t) = |\rho(t)^2| = \left| \int \psi(x, 0) \cdot \psi(x, 0)dx \right|^2. \tag{4}$$

On the other hand, the quantity $\rho(t)$ can be represented as a scalar product of the function expansion coefficients $\psi(x, 0)$ in Eq. (2) and $\psi(x, t)$ in Eq. (3)

$$\rho(t) = \int e^{-\frac{iEt}{\hbar}} C(E) \cdot C(E)dE. \tag{5}$$

In Eq. (5) functions $C(E)$ describe energy distribution for the initial state and any other moment of time

$$|C(E)|^2 = W(E)dE = dW(E). \tag{6}$$

This leads to another quantity $\rho(t)$form in Eq. (6)

$$\rho(t) = \int e^{-\frac{iEt}{\hbar}} W(E)dW = \int e^{-\frac{iEt}{\hbar}} dW(E). \tag{7}$$

Thus, the previously introduced probability that by the time t the system remains its state is described by an equation

$$L(t) = |\rho(t)|^2 = \left| \int e^{-\frac{iEt}{\hbar}} dW(E) \right|^2. \tag{8}$$

Consequently, the state decay law ψ_0 depends only on the energy distribution function in this state and is expressed by the Eq. (8). With the appropriate definition of the integral distribution function $W(E)$ Eq. (8) will also be valid in the case when the function $W(E)$ is discontinuous (point spectrum).

As a result one can define the following statement: the law of decay can be the same for two different states, if only the distribution function for them is the same. In this case time

t for the decay probability counts from the moment of the last statement that atom (system) has not yet decayed. The non-decayed ψ_0 state remains unchanged. Such an aspect can be formulated in other way: the atom does not age, but spontaneously decays. This consequence is valid for any decay law, including the exponential one.

From the physical point of view such mathematical inference denotes the difference between the initial state and the subsequent state for small disintegrating systems. This means unsuitability of the Eq. (1) for such cases and requires a more accurate decay law for these small systems.

3. NUCLEI DECAY WITHIN POISSON DISTRIBUTION FORMALISM

From general theoretical considerations and experimental data, it has long been known that the radioactive nuclei number change $n(t)$ as a function of time t should be well described in terms of the Poisson distribution by the formula

$$n(t) = n(0) \cdot e^{-\lambda t}, \tag{9}$$

where $n(0)$ is the nuclei number at the initial moment of time $t = 0$, $\lambda = 1/\tau$ intensity of radioactive decay, τ average nuclei lifetime. An interpretation of Eq. (9) when describing the decay of quasi-stationary states given by Fock in [7] will be needed further on.

Eq. (9) for the Poisson distribution follows from the system of kinetic equations for the probabilities $p_k(t)$ of simultaneous decay of k nuclei

$$\begin{cases} \frac{dp_0}{dt} + \lambda p_0 = 0 \\ \frac{dp_1}{dt} + \lambda p_1 = \lambda p_0 \\ \dots\dots\dots\dots\dots\dots \\ \frac{dp_k}{dt} + \lambda p_k = \lambda p_{k-1} \end{cases} . \tag{10}$$

The solution of the Eq. (10) system is the numerous quantities

$$p_k(t) = (\lambda k)^k \cdot e^{-\lambda k} \tag{11}$$

the sum of which is normalized to one

$$\sum_{k=0}^{\infty} p_k = 1 \tag{12}$$

with an obvious initial condition $p_0(0) = 1$.

The function $p_0(t)$ is a solution to the first equation in system Eq. (10)

$$\frac{dp_0}{dt} + \lambda p_0 = 0 \tag{13}$$

and describes the system probability to remain in the initial state by the time t

$$p_0(t) = e^{-\lambda t}. \tag{14}$$

In Eq. (14) the value of $p_0(t)$ corresponds to the ratio from Eq. (9):

$$\frac{n(t)}{n(0)} = e^{-\lambda t}. \tag{15}$$

In other words, the value of $n(t)$ satisfies the well-known differential equation

$$-dn = \lambda n dt. \tag{16}$$

A characteristic feature and the main drawback of the Poisson distribution is the fact that the intensity of the process λ does not depend on the system particles number. This is a consequence of the fact that the sum index k in Eq. (12) varies from 0 to ∞.

4. NUCLEI DECAY KINETICS WITHIN BINOMIAL DISTRIBUTION FORMALISM

Let the initial system radioactive nuclei number to be finite and equal to n. In the case of a binomial distribution, the kinetic equations for probabilities ${}_np_k$ of the simultaneous decay of k nuclei are

$$\begin{cases} \frac{d_np_0}{dt} + \lambda_n p_0 = 0 \\ \frac{d_np_k}{dt} + \lambda(n-k)_n p_k = \lambda[n-(k-1)]_n p_{k-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{d_np_n}{dt} + \lambda_n p_n = \lambda_n p_{n-1} \end{cases}, \tag{17}$$

where n is the number of radioactive nuclei. In Eq. (17), the product $\lambda(n-k)$ is interpreted as the radioactive decay intensity , which is the average number of decay events per time unit in a system of $(n-k)$ nuclei.

The radioactive nuclei limited number leads to the fact that the radioactive decay intensity is variable, depending on the decayed nuclei number.In some cases, (e.g. n k) this dependence causes inapplicability of the Poisson distribution. In particular, this nuclei-emitter decay has been considered in previous works [4, 5].

By analogy with the Poisson distribution formalism, in the binomial distribution case, it is also necessary to find an expression for the remaining nuclei number by the time t. The obvious asymptotic condition must be satisfied:

$$\begin{matrix} n(t)^b = n(t)^P = n(0) \cdot e^{-\lambda t} \\ n(0) \to \infty \end{matrix}, \tag{18}$$

where $n(t)^P$ is the nuclei numbers in time dependence for the Poisson distribution and $n(t)^b$ is the same quantity for the binomial distribution.

The problem has a simple solution when using the obvious identity in Eq. (19) and explicit form of probabilities ${}_np_k$.

$$\sum_{k=0}^{\infty} {}_np_k = 1. \tag{19}$$

In the case of a binomial distribution, the probability that number of transformations k occur in a system of n nuclei over a period of time is given by the formula

$${}_np_k = C_n^k f^k (1-f)^{n-k}, \tag{20}$$

where $C_n^k = n!/(n-k)!k!$ are binomial coefficients, and f is decay probability of any certain nucleus. Total decay probability for a system of n nuclei is given by Eq. (21) bellow

$$f = \frac{<k>}{n}, \tag{21}$$

where $<k>$ is an average number of nuclei (from total n nuclei) experiencing the transformation in a given period of time. Relation in Eq. (21) can also be obtained using the formula for the ratio k/n average value in the sum of a series with binomial coefficients [**?**]

$$\sum_{k=0}^{\infty} \frac{<k>}{n} \cdot C_n^k f^k (1-f)^{n-k} = x. \tag{22}$$

Assuming Eq. (20)) and Eq. (22) one can derive probability ${}_np_k$ from Eq. (17) in following form

$${}_np_k = C_n^k \left(\frac{<k>}{n}\right)^k \left(1 - \frac{<k>}{n}\right)^{n-k}. \tag{23}$$

Using the expression for ${}_np_k$ in Eq. (23), one can find the function $n(t)$ from Eq. (18) describing the number of nuclei remaining by the time t in the case of the binomial distribution. It is necessary to represent the expression in Eq. (19) as given below

$$\sum_{n-k=0}^{n} {}_np_k = [f + (1-f)]^n \equiv 1, \tag{24}$$

and then present it in detail to achieve a required result:

$$\begin{array}{c} \sum_{k=0}^{n} {}_np_k = C_n^0 f^n + C_n^1 f^{n-1}(1-f) + \ldots \\ +C_n^{n-k} f^k (1-f)^{n-k} + \ldots \\ +C_n^n (1-f)^n \end{array} . \tag{25}$$

It is obvious, why Eq. (25) is connected to binomial distribution.

In a case of a system with radioactive nuclei the quantity $f =< k > /n$ from Eq. (21) depends on nuclei number n. If Eq. (21) is taken into account, then Eq. (25) can be presented in the following form

$$\begin{array}{c} \sum_{k=0}^{n} {}_np_k = C_n^0 \left(\frac{<k>}{n}\right)^n + C_n^1 \left(\frac{<k>}{n}\right)^{n-1} \left(1 - \frac{<k>}{n}\right) + \ldots \\ +C_n^{n-k} \left(\frac{<k>}{n}\right)^k \left(1 - \frac{<k>}{n}\right)^{n-k} + \ldots \\ +C_n^n \left(1 - \frac{<k>}{n}\right)^n \end{array} . \tag{26}$$

Same equation was presented in [4].

From the structure of terms in the Eq. (26) it can be seen that in the case of binomial distribution the function below befits the same one in Eq. (15)

$$n(t)^b = n \cdot C_n^n \left(1 - \frac{< k >}{n}\right)^n = n \cdot \left(1 - \frac{< k >}{n}\right)^n , \tag{27}$$

where $C_n^n = 1$. For the case of $n \to \infty$ the function $n(t)^b$ is converted to exponential.

It is well known that exponential function can be considered as a power function limit [10]:

$$e = \lim \left(1 + \frac{1}{s}\right)^s , e^x = \lim \left(1 + \frac{x}{s}\right)^s , e^{-x} = \lim \left(1 - \frac{x}{s}\right)^s , \tag{28}$$

So for the asymptotic limit of function $n(t)^b$ in Eq. (18) for a large number of nuclei is to be

$$n(t)^b_{as} = n \cdot e^{-<k>}. \tag{29}$$

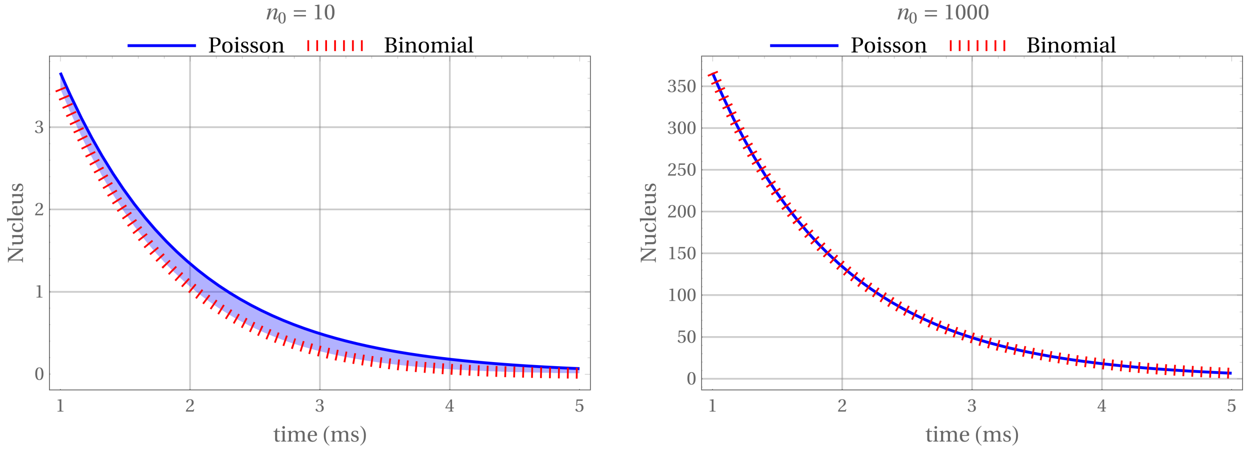
As $n(t)^b_{as}$ in Eq. (29) must equal $n(t)$ in Eq. (15) on asymptotic the following equation for quantity $< k >$ can be obtained:

$$\langle k(t) \rangle = \lambda t. \tag{30}$$

Hence the function in Eq. (27) for an average number of nuclei in the system by the moment of time t in case of binomial distribution takes following form

$$n(t)^b = n \cdot \left(1 - \frac{\lambda t}{n}\right)^n . \tag{31}$$

The form of function $n(t)^b$ in Eq. (31) can be derived in another way as well. It is possible by obtaining Poisson distribution sum $\sum p_k = 1$ from binomial distribution sum $\sum_n p_k = 1$ by making the transition $n \to \infty$. Then the following equation is true for any ${}_np_k$ quantity on asymptotic



(a) 10 abstract nuclei

(b) 1000 abstract nuclei

FIG. 1: Two decay laws comparison for a system with 10 (a) and 1000 (b) with abstract nuclei with short lifetime.

$$\begin{array}{r} {}_{n}p_k|_{n\to\infty} = C_n^{n-k} \left(\frac{<k>}{n}\right)^k \left(1 - \frac{<k>}{n}\right)^{n-k} = \\ = \left[\frac{C_n^{n-k}}{n^k}\right] < k >^k e^{-<k>} = \\ = \left[\frac{n!}{k!(n-k)!n^k}\right] < k >^k e^{-<k>} \to \frac{<k>^k e^{-<k>}}{k!} \end{array} . \tag{32}$$

This means ${}_{n}p_k \to p_k$ for $n \to \infty$ with global term for any ${}_{n}p_k$

$$\left(1 - \frac{< k >}{n}\right)^{n-k}\Bigg|_{n\to\infty} \to e^{-<k>}, \tag{33}$$

which plays role of quantity $p_0(t) = e^{-\lambda t}$ in Eq. (14) for Poisson distribution.

The function $n(t)^b$ in Eq. (31) is more general than the function $n(t)^b_{as}$ in Eq. (29). Yet from the practical point of view function $n(t)^b$ from Eq. (31) is much easier to work with. Therefore, it makes sense to establish the limits of applicability. One can represent function $n(t)^b$ from Eq. (31) in the same form as in Eq. (29):

$$n(t)^b = n \cdot \exp\left[n \cdot \ln\left(1 - \frac{\lambda t}{n}\right)\right]. \tag{34}$$

If the following condition is met for Eq. (34)

$$\frac{\lambda t}{n} << 1 \tag{35}$$

then

$$n \cdot \ln\left(1 - \frac{\lambda t}{n}\right) \approx -\lambda t - \frac{(\lambda t)^2}{2n} - \frac{(\lambda t)^3}{2n^2} - \cdots . \tag{36}$$

And the same function on asymptotic ($n \to \infty$):

$$n \cdot \ln\left(1 - \frac{\lambda t}{n}\right)\Bigg|_{n\to\infty} = -\lambda t, \tag{37}$$

which refers to the same function $n(t)^b_{as}$ in Eq. (31).

Never the less for the case of $\lambda t \approx n$ the more accurate function $n(t)^b$ from Eq. (31) or Eq. (34) must be applied. The difference between the Poisson and binomial approaches can be recognized in a comparison two decay laws of the system with 10 and 1000 abstract nuclei with a short lifetime value ($\lambda = 0.69$ms) as an example shown on figure 1.

As can be seen on figure the difference in approach is essential for a fast decaying systems with a small amount of nuclei in it. Hence, it is important to use a suitable distribution law for describing such systems.

Conclusion

As it was shown in the article the certain difference takes place between two decay laws when one describes a small-scaled decaying systems. To improve the mathematical part of the radioactive decay law the Fock's quasi-stationary theorem statements were used for the description of the decay process physical part. Then the Poisson and binomial distributions were considered to described the nucleus decay for a small-scale system case. The test calculations of the problem were made for the abstract atoms of 10 and 1000 amount with a very low lifetime value ($\lambda = 0.69$ms). The difference between two approaches is clear for that case and must be considered, when it is is necessary. More evaluations must be made for further improvement of theoretical analysis and predictions for the nuclear physics generally.

However, despite the benefits, the calculation of power distribution and flux Monte Carlo requires a lot of computing resources and time. It is therefore necessary to decide on the feasibility of using the Monte Carlo method for the solution of a problem.

1 T.N. Korbut et al., Bull. Russ. Acad. Sci.: Phys. **79** (2015) 461–469.
2 M.O. Kravchenko et al., J.Phys.: Conf. ser. **1133** (2018) 012023.
3 D. Kendall, Ann. Math. Statist. **15** (1948) 1–15.
4 T.N. Korbut et al., Bull. Russ. Acad. Sci.: Phys. **82** (2018) 80–86.
5 M.O. Kravchenko et al., EPJ.: Web conf. **201** (2019) 08004.
6 E. Serge, Wxperimental nuclear pysics (Wiley, 1959)
7 V. Fock, Fundamentals of Quantum Mechanics. Translation from the 2-nd edition. (Mir Publishers, Moscow, 1983) 367.
8 A.S. Davydov, Quantum mechanics. (Pergamin Press, New York: Oxford, 1976) 652.
9 V. Danilov, Matematicheskij analiz. Funkcii, predely, rjady, cepnye drobi (Moscow, Fizmatgiz, 1963) 360.
10 R.S. Guter, Jelementy teorii funkcij. Funkcii dejstvitel'nogo peremennogo. Priblizhenie funkcij. Pochti-periodicheskie funkcii (Moscow, Fizmatgiz, 1963) 248.