Solving Spinor Maxwell Equations in Cylindric Parabolic Coordinates

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Maxwell equation in any Riemannian space-time can be presented in spinor form, on the base of tetrad method by Tetrode-Weyl-Fock-Ivanenko, when the Maxwell field is described by local 2-nd rank symmetrical spinor. This general covariant equation is specified in cylindric parabolic coordinates and corresponding diagonal tetrad. After separating the variables, we derive the system of four 1-st order differential equations in partial derivatives for three functions, depending on two parabolic coordinates. The mathematical task reduces to one 2-nd order equation in partial derivative for a main function, which determining all remaining functions. Solutions are constructed in terms of the confluent hypergeometric functions.

It is shown that diagonalization of the helicity operator for 2-rank symmetric spinor it follows the system of equations coincided with that following from the Maxwell equation, when identifying the eigenvalue σ with the frequency ω. An this fact does not depend on the choice of coordinates and tetrad, Cartesian or cylindric parabolic.

I. SPINOR FORM OF MAXWELL EQUATIONS

To introduce spinor notations, let us start with the ordinary Dirac equation [1]

$$(i\gamma^a \partial_a - m)\Psi = 0, \gamma^a = \begin{vmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{vmatrix}, \Psi = \begin{vmatrix} \xi^\alpha \\ \eta_{\dot{\alpha}} \end{vmatrix}, \{\alpha, \dot{\alpha}\} = 1, 2\,; \tag{1}$$

$\sigma^a = (I, \sigma^j),\ \bar{\sigma}^a = (I, -\sigma^j)$. In 2-spinor form we have two equations

$$i\sigma^a \partial_a \xi = m\eta\,, \quad i\bar{\sigma}^a \partial_a \eta = m\xi\,. \tag{2}$$

It is convenient to attach spinor indices to Pauli matrices: $\sigma^a = (\sigma^a)_{\dot{\beta}\alpha}\,,\ \bar{\sigma}^a = (\bar{\sigma}^a)^{\beta\dot{\alpha}}$, then eqs. (2) read

$$i(\sigma^a \partial_a)_{\dot{\beta}\alpha}\, \xi^\alpha = m\eta_{\dot{\beta}}, \quad i(\bar{\sigma}^a \partial_a)^{\beta\dot{\alpha}}\, \eta_{\dot{\alpha}} = m\xi^\beta\,. \tag{3}$$

Electromagnetic tensor is equivalent to a pair of symmetrical 2-rang spinors: $F_{mn} \longleftrightarrow \{\xi^{\alpha\beta}, \eta_{\dot{\alpha}\dot{\beta}}\}$; correspondingly, 8 Maxwell equations are presented as follows

$$(\sigma^a \partial_a)_{\dot{\rho}\alpha}\, \xi^{\alpha\beta} = (\sigma^b)_{\dot{\rho}\alpha} \omega^{\alpha\beta}\, J_b\,, \quad (\bar{\sigma}^a \partial_a)^{\rho\dot{\alpha}}\, \eta_{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}^b)^{\rho\dot{\alpha}} \omega_{\dot{\alpha}\dot{\beta}}\, J_b\,; \tag{4}$$

the second equation is conjugate to the first. In (4) we use spinor metrical matrices [1]

$$(\epsilon_{\alpha\beta}) = i\sigma^2\,, \quad (\epsilon^{\alpha\beta}) = -i\sigma^2\,; \quad (\epsilon_{\dot{\alpha}\dot{\beta}}) = i\sigma^2\,, \quad (\epsilon^{\dot{\alpha}\dot{\beta}}) = -i\sigma^2\,. \tag{5}$$

To prove equivalence of the spinor form (4) to ordinary Maxwell equation in vector notations let us apply notations without spinor indices. To this end, we take into account identities [1]

$$(\xi^{\alpha\beta}) = \Sigma^{mn} F_{mn} \sigma^2\,,\ (\eta_{\dot{\alpha}\dot{\beta}}) = -\bar{\Sigma}^{mn} F_{mn} \sigma^2\,,\ \Sigma^{mn} = \frac{1}{4}(\bar{\sigma}^m \sigma^n - \bar{\sigma}^n \sigma^m)\,,\ \bar{\Sigma}^{mn} = \frac{1}{4}(\sigma^m \bar{\sigma}^n - \sigma^n \bar{\sigma}^m)\,. \tag{6}$$

Then, eqs. (4) may be re-written as

$$\sigma^a \partial_a\, \Sigma^{mn} F_{mn} = -\sigma^b J_b\,, \qquad \bar{\sigma}^a \partial_a\, \bar{\Sigma}^{mn} F_{mn} = -\bar{\sigma}^b J_b\,. \tag{7}$$

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We are to take into account identities

$$\Sigma^{mn} F_{mn} = \sigma^1 (F_{01} - iF_{23}) + \sigma^2 (F_{02} - iF_{31}) + \sigma^3 (F_{03} - iF_{12}) \, ,$$
$$\bar{\Sigma}^{mn} F_{mn} = \sigma^1 (-F_{01} - iF_{23}) + \sigma^2 (-F_{02} - iF_{31}) + \sigma^3 (-F_{03} - iF_{12}) \, ;$$

with notations

$$F_{01} = -E^1 \, , F_{02} = -E^2 \, , F_{03} = -E^3 \, , F_{23} = B^1 \, , F_{31} = B^2 \, , F_{12} = B^3 \tag{8}$$

they read

$$\Sigma^{mn} F_{mn} = -\sigma^1 (E^1 + iB^1) - \sigma^1 (E^2 + iB^2) - \sigma^1 (E^3 + iB^3) = -\sigma^j a_j ,$$
$$\bar{\Sigma}^{mn} F_{mn} = \sigma^1 (E^1 - iB^1) + \sigma^1 (E^2 - iB^2) + \sigma^1 (E^3 - iB^3) = +\sigma^j b_j , \tag{9}$$

and

$$(\xi^{\alpha\beta}) = \Sigma^{mn} F_{mn} \sigma^2 = \begin{vmatrix} -i(a_1 - ia_2) & ia_3 \\ ia_3 & +i(a_1 + ia_2) \end{vmatrix} , \; (\eta_{\dot{\alpha}\dot{\beta}}) = -\bar{\Sigma}^{mn} F_{mn} \sigma^2 = \begin{vmatrix} -i(b_1 - ib_2) & ib_3 \\ ib_3 & i(b_1 + ib_2) \end{vmatrix} .$$

Taking into account (9), Maxwell equations (7) may be presented in the form

$$(\partial_0 + \sigma^l \partial_l) \, (\sigma^k a_k) = J_0 + \sigma^j J_j \, , \quad (\partial_0 - \sigma^l \partial_l) \, (\sigma^k b_k) = -J_0 + \sigma^j J_j \, . \tag{10}$$

From (10) we derive

$$\sigma^n \partial_0 a_n + (\delta_{lk} + i\omega_{nlk} \sigma^n) \partial_l a_k = J_0 + \sigma^n J_n , \quad \sigma^n \partial_0 b_n - (\delta_{lk} + i\omega_{nlk} \sigma^n) \partial_l b_k = -J_0 + \sigma^n J_n \, .$$

Therefore, we have four equations

$$\partial_l a_l = J_0 \, , \quad \partial_0 a_n + i\omega_{nlk} \partial_l a_k = J_n \, , \quad \partial_l b_l = J_0 \, , \quad \partial_0 b_n - i\omega_{nlk} \partial_l b_k = J_n \, ,$$

or differently

$$(1) \quad \partial_l (E^l + iB^l) = J_0 \, , \qquad (2) \quad \partial_0 (E^l + iB^l) + i\omega_{nlk} \partial_l (E^k + iB^k) = J_n \, ,$$
$$(1') \quad \partial_l (E^l - iB^l) = J_0 \, , \qquad (2') \quad \partial_0 (E^l - iB^l) - i\omega_{nlk} \partial_l (E^k - iB^l) = J_n \, .$$

Summing and subtracting equations within each pair, we obtain

$$1 + 1', \quad \partial_l E^l = J_0 \, , \quad 1 - 1', \quad \partial_l B^l = 0 \, ,$$
$$2 + 2', \quad \partial_0 E^n - \omega_{nlk} \partial_l B^k = J_k \, , \quad 2 - 2', \quad \partial_0 B^n + \omega_{nlk} \partial_l E^k = 0 \, ;$$

they may be identified with Maxwell equations in vector form

$$\text{div} \, \mathbf{E} = J^0 , \quad \text{div} \, \mathbf{B} = 0 \, , \quad \text{rot} \, \mathbf{B} = \partial_0 \mathbf{E} + \mathbf{J} \, , \quad \text{rot} \, \mathbf{E} = -\partial_0 \mathbf{B} \, , \tag{11}$$

where $\mathbf{E} = (E^n) \, , \; \mathbf{B} = (B^n) \, , \quad J^0 = J_0 \, , \; \mathbf{J} = (J^n) = (-J_n)$.

II. MAXWELL EQUATIONS IN CYLINDRIC PARABOLIC COORDINATES

Let us construct solutions of spinor Maxwell equations in cylindric parabolic coordinates

$$x_1 = \frac{u^2 - v^2}{2} \, , \quad x_2 = uv \, , \quad x_3 = z \, ; \tag{12}$$

$$v = +\sqrt{-x_1 + \sqrt{x_1^2 + x_2^2}} \, , \quad u = \pm\sqrt{+x_1 + \sqrt{x_1^2 + x_2^2}} \, . \tag{13}$$

The metric of Minkowski space takes the form (let $x^\alpha = (t, u, v, z)$)

$$dS^2 = dt^2 - (u^2 + v^2)(du^2 + dv^2) - dz^2 \, , \quad g_{\alpha\beta} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -(u^2 + v^2) & 0 & 0 \\ 0 & 0 & -(u^2 + v^2) & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} . \tag{14}$$

Below we will use the diagonal tetrad (recall $g_{\alpha\beta}(x) e_{(k)}^{\ \alpha} e_{(l)}^{\ \beta} = \eta_{kl}$)

$$e_{(k)}^{\ \alpha} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{u^2+v^2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{u^2+v^2} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| , \qquad e_{(k)\alpha} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -\sqrt{u^2+v^2} & 0 & 0 \\ 0 & 0 & -\sqrt{u^2+v^2} & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| . \tag{15}$$

In order to find Ricci rotation coefficients, let us introduce auxiliary quantities (see in [2]): $\lambda_{abc} = \gamma_{abc} - \gamma_{acb}$. For λ_{abc} we easily derive the following representation

$$\lambda_{abc} = \gamma_{abc} - \gamma_{acb} = (e_{(a)\alpha;\beta} - e_{(a)\beta;\alpha})\, e_{(c)}^{\alpha} e_{(b)}^{\beta} = (\partial_\beta e_{(a)\alpha} - \Gamma^{\rho}_{\alpha\beta} e_{(a)\rho} - \partial_\alpha e_{(a)\beta} + \Gamma^{\rho}_{\beta\alpha} e_{(a)\rho})\, e_{(c)}^{\alpha} e_{(b)}^{\beta} \, ;$$

that is

$$\lambda_{abc} = [\partial_\beta e_{(a)\alpha} - \partial_\alpha e_{(a)\beta}]\, e_{(c)}^{\alpha} e_{(b)}^{\beta} \, ; \tag{16}$$

according to (16), λ_{abc} are calculated with the use of ordinary derivatives. Besides, there exists an identity

$$\frac{1}{2}(\lambda_{abc} + \lambda_{bca} - \lambda_{cab}) \equiv \frac{1}{2}(\gamma_{abc} - \gamma_{acb} + \gamma_{bca} - \gamma_{bac} - \gamma_{cab} + \gamma_{cba}) \equiv \gamma_{abc} \, . \tag{17}$$

We need explicit expressions for λ_{abc}. First of all, we have two relations

$$a = 0, \quad \lambda_{0bc} = 0, \qquad a = 3, \quad \lambda_{0bc} = 0 \, .$$

Further, taking in mind diagonal structure of the tetrad, we derive the formula

$$\lambda_{1bc} = e_{(b)}^{\beta} e_{(c)}^{1} \cdot \partial_\beta e_{(1)1} - e_{(b)}^{1} e_{(c)}^{\beta} \cdot \partial_\beta e_{(1)1} =$$
$$= e_{(b)}^{1} e_{(c)}^{1} \cdot \partial_1 e_{(1)1} + e_{(b)}^{2} e_{(c)}^{1} \cdot \partial_2 e_{(1)1} - e_{(b)}^{1} e_{(c)}^{1} \cdot \partial_1 e_{(1)1} - e_{(b)}^{1} e_{(c)}^{2} \cdot \partial_2 e_{(1)1} \, ,$$

that is

$$\lambda_{1[bc]} = [e_{(b)}^{2} e_{(c)}^{1} - e_{(b)}^{1} e_{(c)}^{2}]\, \partial_2 e_{(1)1} = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -e_{(1)}^{1} e_{(2)}^{2} & 0 \\ 0 & e_{(1)}^{1} e_{(2)}^{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| \partial_2 e_{(1)1} \, ;$$

and similarly

$$\lambda_{2bc} = e_{(b)}^{\beta} e_{(c)}^{2} \cdot \partial_\beta e_{(2)2} - e_{(b)}^{2} e_{(c)}^{\beta} \cdot \partial_\beta e_{(2)2} =$$
$$= e_{(b)}^{1} e_{(c)}^{2} \cdot \partial_1 e_{(2)2} + e_{(b)}^{2} e_{(c)}^{2} \cdot \partial_2 e_{(2)2} - e_{(b)}^{2} e_{(c)}^{1} \cdot \partial_1 e_{(2)2} - e_{(b)}^{2} e_{(c)}^{2} \cdot \partial_2 e_{(2)2} \, ,$$

so that

$$\lambda_{2[bc]} = [e_{(b)}^{1} e_{(c)}^{2} - e_{(b)}^{2} e_{(c)}^{1}]\, \partial_1 e_{(2)2} = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & e_{(1)}^{1} e_{(2)}^{2} & 0 \\ 0 & -e_{(1)}^{1} e_{(2)}^{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| \partial_1 e_{(2)2} \, .$$

Allowing for relations $e_{(1)}^{1} = e_{(2)}^{2} = 1/\sqrt{u^2+v^2}$, $e_{(1)1} = e_{(2)2} = -\sqrt{u^2+v^2}$, we obtain needed formulas for coefficients $\lambda_{1[bc]}$ $\lambda_{2[bc]}$:

$$\lambda_{1[bc]} = \frac{1}{(u^2+v^2)^{3/2}} \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & +v & 0 \\ 0 & -v & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| , \qquad \lambda_{2[bc]} = \frac{1}{(u^2+v^2)^{3/2}} \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -u & \\ 0 & +u & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| . \tag{18}$$

Thus, nonvanishing are only the following coefficients

$$\lambda_{1[12]} = +\frac{v}{(u^2+v^2)^{3/2}} , \qquad \lambda_{1[21]} = -\frac{v}{(u^2+v^2)^{3/2}} ,$$
$$\lambda_{2[12]} = -\frac{u}{(u^2+v^2)^{3/2}} , \qquad \lambda_{1[21]} = +\frac{u}{(u^2+v^2)^{3/2}} . \tag{19}$$

Now, when using the formula $\gamma_{[ab]c} = \frac{1}{2}(-\lambda_{c[ab]} + \lambda_{a[bc]} - \lambda_{b[ac]})$, it is convenient to split it into four cases

$$\gamma_{[ab]0} = \frac{1}{2}(-\lambda_{0[ab]} + \lambda_{a[b0]} - \lambda_{b[a0]})\,, \quad \gamma_{[ab]3} = \frac{1}{2}(-\lambda_{3[ab]} + \lambda_{a[b3]} - \lambda_{b[a3]})\,,$$
$$\gamma_{[ab]1} = \frac{1}{2}(-\lambda_{1[ab]} + \lambda_{a[b1]} - \lambda_{b[a1]})\,, \quad \gamma_{[ab]2} = \frac{1}{2}(-\lambda_{2[ab]} + \lambda_{a[b2]} - \lambda_{b[a2]})\,.$$

we find nonvanishing Ricci coefficients

$$\gamma_{[12]1} = -\gamma_{[21]1} = -\lambda_{1[12]} = -\frac{v}{(u^2+v^2)^{3/2}}\,, \quad \gamma_{[12]2} = -\gamma_{[21]2} = -\lambda_{2[12]} = +\frac{u}{(u^2+v^2)^{3/2}}\,. \tag{20}$$

Now let us turn to Maxwell equations in spinor form, when using the tetrad formalism:

$$\left[\sigma^c e^{\alpha}_{(c)}(x)\partial_\alpha + \sigma^c(\frac{1}{2}\Sigma^{ab}\otimes I + I\otimes\frac{1}{2}\Sigma^{ab})\gamma_{[ab]c}(x)\right]\xi(x) = 0\,, \tag{21}$$

where $\Sigma^{0j} = \frac{1}{2}\sigma^j,\ \Sigma^{12} = -\frac{i}{2}\sigma^3,\ \Sigma^{23} = -\frac{i}{2}\sigma^1,\ \Sigma^{31} = -\frac{i}{2}\sigma^2$. With eqs. (20), eq. (21) takes the form

$$\Big[\sigma^0\partial_t + \sigma^3\partial_z + \sigma^1 e^1_{(1)}(x)\partial_u + \sigma^2 e^2_{(2)}(x)\partial_v +$$
$$+\sigma^1(\Sigma^{12}\otimes I + I\otimes\Sigma^{12})\gamma_{[12]1} + \sigma^2(\Sigma^{12}\otimes I + I\otimes\Sigma^{12})\gamma_{[12]2}\Big]\,\xi(x) = 0\ ,$$

or differently

$$\Big\{\sigma^0\partial_t + \sigma^3\partial_z + \frac{1}{\sqrt{u^2+v^2}}\left(\sigma^1\partial_u + \sigma^2\partial_v\right) +$$
$$+\frac{i/2}{(u^2+v^2)^{3/2}}\left[v\,\sigma^1(\sigma^3\otimes I + I\otimes\sigma^3) - u\,\sigma^2(\sigma^3\otimes I + I\otimes\sigma^3)\right]\Big\}\,\xi(x) = 0\,. \tag{22}$$

We use the following substitution for electromagnetic symmetric 2-rank spinor $\xi(x)$:

$$\xi(x) = e^{-i\omega t}e^{ikz}\left|\begin{array}{cc} f(u,v) & h(u,v) \\ h(u,v) & g(u,v)\end{array}\right|, \tag{23}$$

where f, g, h stand for some functions over the variables u, v. taking into account expressions for Pauli matrices

$$\sigma^0 = \left|\begin{array}{cc} 1 & 0 \\ 0 & 1\end{array}\right|, \quad \sigma^1 = \left|\begin{array}{cc} 0 & 1 \\ 1 & 0\end{array}\right|, \quad \sigma^2 = \left|\begin{array}{cc} 0 & -i \\ i & 0\end{array}\right|, \quad \sigma^3 = \left|\begin{array}{cc} 1 & 0 \\ 0 & -1\end{array}\right|,$$

from (22) we derive

$$-i\omega\left|\begin{array}{cc} f & h \\ h & g\end{array}\right| + ik\left|\begin{array}{cc} f & h \\ -h & -g\end{array}\right| + \frac{1}{\sqrt{u^2+v^2}}\partial_u\left|\begin{array}{cc} h & g \\ f & h\end{array}\right| + \frac{1}{\sqrt{u^2+v^2}}\partial_v\left|\begin{array}{cc} -ih & -ig \\ if & ih\end{array}\right| +$$
$$+\frac{i/2}{(u^2+v^2)^{3/2}}\left|\begin{array}{cc} 0 & -2vg \\ 2vf & 0\end{array}\right| + \frac{i/2}{(u^2+v^2)^{3/2}}\left|\begin{array}{cc} 0 & -2iug \\ -2iuf & 0\end{array}\right| = \left|\begin{array}{cc} 0 & 0 \\ 0 & 0\end{array}\right|,$$

or

$$-i\omega\left|\begin{array}{cc} f & h \\ h & g\end{array}\right| + ik\left|\begin{array}{cc} f & h \\ -h & -g\end{array}\right| + \frac{1}{\sqrt{u^2+v^2}}\left|\begin{array}{cc} (\partial_u - i\partial_v)h & (\partial_u - i\partial_v)g \\ (\partial_u + i\partial_v)f & (\partial_u + i\partial_v)h\end{array}\right| +$$
$$+\frac{i}{(u^2+v^2)^{3/2}}\left|\begin{array}{cc} 0 & -(v+iu)g \\ (v-iu)f & 0\end{array}\right| = \left|\begin{array}{cc} 0 & 0 \\ 0 & 0\end{array}\right|,$$

Whence follow four differential equations

$$\begin{aligned}
(11)\quad & -i\omega f + ikf + \frac{1}{\sqrt{u^2+v^2}}(\partial_u - i\partial_v)h = 0\,,\\
(22)\quad & -i\omega g - ikg + \frac{1}{\sqrt{u^2+v^2}}(\partial_u + i\partial_v)h = 0\,,\\
(12)\quad & -i\omega h + ikh + \frac{1}{\sqrt{u^2+v^2}}(\partial_u - i\partial_v)g - \frac{i(v+iu)}{(u^2+v^2)^{3/2}}g = 0\,,\\
(21)\quad & -i\omega h - ikh + \frac{1}{\sqrt{u^2+v^2}}(\partial_u + i\partial_v)f + \frac{i(v-iu)}{(u^2+v^2)^{3/2}}f = 0\,.
\end{aligned} \tag{24}$$

They may be re-written differently

$$
\begin{aligned}
-i\omega f + ikf + \frac{1}{\sqrt{u^2+v^2}}\partial_u h - \frac{i}{\sqrt{u^2+v^2}}\partial_v h = 0\,,\\
-i\omega g - ikg + \frac{1}{\sqrt{u^2+v^2}}\partial_u h + \frac{i}{\sqrt{u^2+v^2}}\partial_v h = 0\,,\\
-i\omega h + ikh + \frac{1}{\sqrt{u^2+v^2}}\left(\partial_u + \frac{u}{u^2+v^2}\right) g - \frac{i}{\sqrt{u^2+v^2}}\left(\partial_v + \frac{v}{u^2+v^2}\right) g = 0\,,\\
-i\omega h - ikh + \frac{1}{\sqrt{u^2+v^2}}\left(\partial_u + \frac{u}{u^2+v^2}\right) f + \frac{i}{\sqrt{u^2+v^2}}\left(\partial_v + \frac{v}{u^2+v^2}\right) f = 0\,.
\end{aligned}
$$

Taking into account the two identities for function g:

$$
\begin{aligned}
\frac{1}{\sqrt{u^2+v^2}}\left(\partial_u + \frac{u}{u^2+v^2}\right) g = \frac{1}{u^2+v^2}\partial_u \sqrt{u^2+v^2} g, \quad \sqrt{u^2+v^2} g \equiv \bar{g},\\
\frac{1}{\sqrt{u^2+v^2}}\left(\partial_v + \frac{v}{u^2+v^2}\right) g = \frac{1}{u^2+v^2}\partial_v \sqrt{u^2+v^2} g, \quad \sqrt{u^2+v^2} g \equiv \bar{g},
\end{aligned}
$$

and similar ones for f:

$$
\begin{aligned}
\frac{1}{\sqrt{u^2+v^2}}\left(\partial_u + \frac{u}{u^2+v^2}\right) f = \frac{1}{u^2+v^2}\partial_u \sqrt{u^2+v^2} f, \quad \sqrt{u^2+v^2} f \equiv \bar{f},\\
\frac{1}{\sqrt{u^2+v^2}}\left(\partial_v + \frac{v}{u^2+v^2}\right) f = \frac{1}{u^2+v^2}\partial_v \sqrt{u^2+v^2} f, \quad \sqrt{u^2+v^2} f \equiv \bar{f},
\end{aligned}
$$

and introducing new notations

$$
\sqrt{u^2+v^2} f \equiv \bar{f}, \quad \sqrt{u^2+v^2} g \equiv \bar{g}, \tag{25}
$$

we present the above system of four equations as follows

$$
\begin{aligned}
-i\omega \bar{f} + ik\bar{f} + \partial_u h - i\partial_v h = 0\,, \quad -i\omega \bar{g} - ik\bar{g} + \partial_u h + i\partial_v h = 0\,,\\
-i\omega h + ikh + \frac{1}{u^2+v^2}\partial_u \bar{g} - \frac{i}{u^2+v^2}\partial_v \bar{g} = 0\,,\\
-i\omega h - ikh + \frac{1}{u^2+v^2}\partial_u \bar{f} + \frac{i}{u^2+v^2}\partial_v \bar{f} = 0\,.
\end{aligned} \tag{26}
$$

Let us sum and subtract equations within each pair, the with the notations

$$
\bar{f} + \bar{g} = F\,, \quad \bar{f} - \bar{g} = G\,, \qquad F + G = 2\bar{f}\,, \quad F - G = 2\bar{g}\,; \tag{27}
$$

we obtain the following system

$$
\begin{aligned}
-i\omega F + ikG + 2\partial_u h = 0\,, \quad -i\omega G + ikF - 2i\partial_v h = 0\,,\\
-2i\omega h + \frac{1}{u^2+v^2}\partial_u F + \frac{i}{u^2+v^2}\partial_v G = 0\,, \quad 2ikh - \frac{1}{u^2+v^2}\partial_u G - \frac{i}{u^2+v^2}\partial_v F = 0\,.
\end{aligned} \tag{28}
$$

From two first equations we can find expressions for F, G:

$$
F = \frac{2}{\omega^2 - k^2}[-i\omega\partial_u h - k\partial_v h]\,, \quad G = \frac{2}{\omega^2 - k^2}[-ik\partial_u h - \omega\partial_v h]\,, \tag{29}
$$

substituting them into remaining two equations we get

$$
\begin{aligned}
-2i\omega h + \frac{1}{u^2+v^2}\frac{2}{\omega^2-k^2}\left\{\partial_u[-i\omega\partial_u h - k\partial_v h] + i\partial_v[-ik\partial_u h - \omega\partial_v h]\right\} = 0\,,\\
2ikh - \frac{1}{u^2+v^2}\frac{2}{\omega^2-k^2}\left\{\partial_u[-ik\partial_u h - \omega\partial_v h] + i\partial_v[-i\omega\partial_u h - k\partial_v h]\right\} = 0\,,
\end{aligned}
$$

or differently

$$-2i\omega h + \frac{1}{u^2+v^2}\frac{2}{\omega^2-k^2}\left\{-i\omega\partial_u^2 - k\partial_{uv}^2 + k\partial_{uv}^2 - i\omega\partial_v^2\right\} h = 0\,,$$
$$2ikh - \frac{1}{u^2+v^2}\frac{2}{\omega^2-k^2}\left\{-ik\partial_u^2 - \omega\partial_{uv}^2 + \omega\partial_{uv}^2 - ik\partial_v^2\right\} h = 0\,.$$

Whence, after simple regrouping the terms follow two coinciding equations

$$-2i\omega h + \frac{1}{u^2+v^2}\frac{2}{\omega^2-k^2}\left\{-i\omega(\partial_u^2+\partial_v^2) - k\partial_{uv}^2 + k\partial_{uv}^2\right\} h = 0\,,$$
$$2ikh - \frac{1}{u^2+v^2}\frac{2}{\omega^2-k^2}\left\{-ik(\partial_u^2+\partial_v^2) - \omega\partial_{uv}^2 + \omega\partial_{uv}^2\right\} h = 0\,.$$

Thus, we arrive at $h(u,v)$:

$$h + \frac{1}{u^2+v^2}\frac{1}{\omega^2-k^2}(\partial_u^2+\partial_v^2)h = 0\,;$$

which can be presented differently (let $\lambda^2 = \omega^2 - k^2 > 0$)

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \lambda^2 u^2 + \lambda^2 v^2\right) h = 0\,. \tag{30}$$

In this equation, the variables ma be separated by substitution $h(u,v) = U(u)\,V(v)$, so we get

$$\left(\frac{1}{U}\frac{d^2}{dv^2}U + \lambda^2 u^2\right) + \left(\frac{1}{V}\partial_v^2 V + \lambda^2 v^2\right) = 0\,.$$

We find two separate equations

$$\begin{aligned} \frac{1}{U}\partial_u^2 U + \lambda^2 u^2 = -A &\implies \left(\frac{d^2}{du^2} + \lambda^2 u^2 + A\right) U = 0\,,\\ \frac{1}{V}\partial_v^2 V + \lambda^2 v^2 = +A &\implies \left(\frac{d^2}{dv^2} + \lambda^2 v^2 - A\right) V = 0\,. \end{aligned} \tag{31}$$

Let us transform eqs. (31) to new variables:

$$X = u^2,\ (\frac{d^2}{dX^2} + \frac{1}{2X}\frac{d}{dX} + \frac{\lambda^2}{4} + \frac{A}{4X})U = 0\,; \qquad Y = v^2,\ (\frac{d^2}{dY^2} + \frac{1}{2Y}\frac{d}{dY} + \frac{\lambda^2}{4} - \frac{A}{4Y})V = 0\,.$$

These equations belong the confluent hypergeometric type. Their solutions are searched in the form

$$U(X) = X^\alpha e^{\beta X} f(X), \qquad V(Y) = Y^\rho e^{\sigma Y} g(Y)\,; \tag{32}$$

taking in mind the formulas

$$U' = X^\alpha e^{\beta X}(\frac{\alpha}{X}f + \beta f + f')\,, \quad V' = Y^\rho e^{\sigma Y}(\frac{a}{Y}g + \sigma g + g')\,,$$
$$U'' = X^\alpha e^{\beta X}\left[f'' + (\frac{2\alpha}{X} + 2\beta)f' + (\frac{\alpha(\alpha-1)}{X^2} + \frac{2\alpha\beta}{X} + \beta^2)f\right]\,,$$
$$V'' = Y^\rho e^{\sigma Y}\left[g'' + (\frac{2\rho}{Y} + 2\sigma)g' + (\frac{\rho(\rho-1)}{Y^2} + \frac{2\rho\sigma}{Y} + \sigma^2)g\right]\,,$$

we obtain

$$\left[\frac{d^2}{dX^2} + (\frac{2\alpha}{X} + \frac{1}{2X} + 2\beta)\frac{d}{dX} + \frac{2\alpha\beta}{X} + \frac{\beta}{2X} + \frac{A}{4X} + \frac{\alpha(\alpha-1)}{X^2} + \frac{\alpha/2}{X^2} + \beta^2 + \frac{\lambda^2}{4}\right] f(X) = 0\,,$$
$$\left[\frac{d^2}{dY^2} + (\frac{2\rho}{Y} + \frac{1}{2Y} + 2\sigma)\frac{d}{dY} + \frac{2\rho\sigma}{Y} + \frac{\sigma}{2Y} - \frac{A}{4Y} + \frac{\rho(\rho-1)}{Y^2} + \frac{\rho/2}{Y^2} + \sigma^2 + \frac{\lambda^2}{4}\right] g(Y) = 0\,.$$

Imposing restrictions

$$\alpha = 0, +\frac{1}{2}, \quad 2\beta = \pm i\lambda\,; \qquad \rho = 0, +\frac{1}{2}, \quad 2\sigma = \pm i\lambda\,,$$

we get two confluent hypergeometric equations

$$\left[X\frac{d^2}{dX^2} + (2\alpha + \frac{1}{2} + 2\beta X)\frac{d}{dX} + 2\alpha\beta + \frac{\beta}{2} + \frac{A}{4}\right] f(X) = 0\,,$$
$$\left[Y\frac{d^2}{dY^2} + (2\rho + \frac{1}{2} + 2\sigma Y)\frac{d}{dY} + 2\rho\sigma + \frac{\sigma}{2} - \frac{A}{4}\right] g(Y) = 0\,.$$

Without loss of generality let us fix two parameters $2\beta = -i\lambda$, $2\sigma = -i\lambda$, this gives

$$\left[X\frac{d^2}{dX^2} + (2\alpha + 1/2 - i\lambda X)\frac{d}{dX} - i\lambda\alpha - i\lambda/4 + A/4\right] f(X) = 0\,,$$
$$\left[Y\frac{d^2}{dY^2} + (2\rho + 1/2 - i\lambda Y)\frac{d}{dY} - i\lambda\rho - i\lambda/4 - A/4\right] g(Y) = 0\,.$$

After transforming equations to new variables

$$i\lambda X = x = i\lambda u^2, \qquad i\lambda Y = y = i\lambda v^2, \quad A/4i\lambda = \Lambda\,, \tag{33}$$

we get

$$\left[x\frac{d^2}{dx^2} + (2\alpha + 1/2 - x)\frac{d}{dx} - \alpha - 1/4 + \Lambda\right] f(x) = 0\,,$$
$$\left[y\frac{d^2}{dy^2} + (2\rho + 1/2 - y)\frac{d}{dy} - \rho - 1/4 - \Lambda\right] g(y) = 0\,. \tag{34}$$

For each equation, there exist two different possibilities,depending on the choice $\alpha = 0, 1/2$ and $\rho = 0, 1/2$. For definiteness, we take the values $\alpha = 0, \rho = 0$:

$$\left[x\frac{d^2}{dx^2} + (1/2 - x)\frac{d}{dx} - (1/4 - \Lambda)\right] f(x) = 0\,, \quad U(x) = e^{x/2} f(x), \quad x = i\lambda u^2;$$
$$\left[y\frac{d^2}{dy^2} + (1/2 - y)\frac{d}{dy} - (1/4 + \Lambda)\right] g(y) = 0\,, \quad V(y) = e^{y/2} g(y), \quad y = i\lambda v^2. \tag{35}$$

Equations (35) can be identified with canonical for of confluent hypergeometric equation

$$zF'' + (c - z)F' - aF = 0, \qquad c = 1/2, \quad a = 1/4 - \Lambda,\ 1/4 + \Lambda\,. \tag{36}$$

Two linearly independent solutions might be used:

$$F_1 = \Phi(a, c; z), \quad F_2 = z^{1-c}\Phi(a - c + 1, 2 - c; z) \tag{37}$$

Therefore, for two equations in (35) we have respective pairs of solutions:

$$U_1 = e^{x/2}\Phi(\frac{1}{4} - \Lambda, \frac{1}{2}; x), \quad U_2 = e^{x/2}\sqrt{x}\ \Phi(\frac{3}{4} - \Lambda, \frac{3}{2} - c; x), \quad \sqrt{x} = \sqrt{i\lambda}\, u\,; \tag{38}$$

$$V_1 = e^{y/2}\Phi(\frac{1}{4} + \Lambda, \frac{1}{2}; y), \qquad V_2 = e^{y/2}\sqrt{y}\ \Phi(\frac{3}{4} + \Lambda, \frac{3}{2}; y), \quad \sqrt{y} = \sqrt{i\lambda}\, v\,. \tag{39}$$

Let us consider parametrization of the plane x_1, x_2 by coordinate $u, v)$ (see (12)– (13) and Fig. 1):

$$x_1 = \frac{u^2 - v^2}{2}\,, \quad x_2 = uv\,, \quad x_3 = z\,,\ u \in (-\infty, +\infty),\ v \in (0, +\infty). \tag{40}$$

We need solutions which are single-valued function in Euclidean plane. It is evident that functions $U_2(u)$ (see (38)) take on different values in identified points $(u, -v)$ and $(u, +v)$. Therefore, we may use only two types of complete solutions:

$$U_1(x) \cdot V_1(y), \qquad U_1(x) \cdot V_2(y)\,. \tag{41}$$

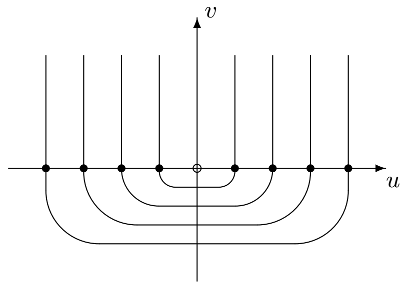

FIG. 1: Identification of the boundary points in $G(u,v)$

It remains unclear the question on physical meaning of separation parameter A. From (31) it follows

$$\hat{A}\,U(u)V(v) = A\,U(u)V(v), \quad \hat{A} = -\frac{1}{2}\left(\frac{\partial^2}{\partial u^2} + \lambda^2 u^2 - \frac{\partial^2}{\partial v^2} - \lambda^2 v^2\right). \qquad (42)$$

Let us find expression for corresponding operator $\hat{A}$ in Cartesian coordinates. Taking into account relations

$$\frac{\partial}{\partial u} = u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial v} = -v\,\frac{\partial}{\partial x} + u\,\frac{\partial}{\partial y},$$

for 2-nd order derivative we find the formulas

$$\frac{\partial^2}{\partial u^2} = u\frac{\partial}{\partial x}u\frac{\partial}{\partial x} + u\frac{\partial}{\partial x}v\frac{\partial}{\partial y} + v\frac{\partial}{\partial y}u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}v\frac{\partial}{\partial y}\,, \quad \frac{\partial^2}{\partial v^2} = v\frac{\partial}{\partial x}v\frac{\partial}{\partial x} - v\frac{\partial}{\partial x}u\frac{\partial}{\partial y} - u\frac{\partial}{\partial y}v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}u\frac{\partial}{\partial y}\,;$$

and further, taking in mind identities

$$\frac{\partial u}{\partial x} = \frac{u}{u^2+v^2}\,, \qquad \frac{\partial u}{\partial y} = \frac{v}{u^2+v^2}\,, \qquad \frac{\partial v}{\partial x} = \frac{-v}{u^2+v^2}\,, \qquad \frac{\partial v}{\partial y} = \frac{u}{u^2+v^2}\,,$$

we derive

$$\frac{\partial^2}{\partial u^2} = u^2\frac{\partial^2}{\partial x^2} + v^2\frac{\partial^2}{\partial y^2} + 2uv\,\frac{\partial^2}{\partial x\partial y} + \frac{\partial}{\partial x}\,, \quad \frac{\partial^2}{\partial v^2} = v^2\frac{\partial^2}{\partial x^2} + u^2\frac{\partial^2}{\partial y^2} - 2uv\,\frac{\partial^2}{\partial x\partial y} - \frac{\partial}{\partial x}\,,$$

that is

$$\frac{1}{2}\,(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}) = \frac{u^2-v^2}{2}\,(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}) + 2uv\,\frac{\partial^2}{\partial x\partial y} + \frac{\partial}{\partial x} = x\,(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}) + 2y\,\frac{\partial^2}{\partial x\partial y} + \frac{\partial}{\partial x}\,. \qquad (43)$$

Therefore, eq. (42) may be presented as

$$-\frac{1}{2}\left[x\,(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2\lambda^2) + 2y\,\frac{\partial^2}{\partial x\partial y} + \frac{\partial}{\partial x}\right]U(u)V(v) = A\,U(u)V(v)\,. \qquad (44)$$

In fact, constructed solutions of the Maxwell equations are numerated with the help of three parameters $\xi_{\omega,k,A}(t,u,v,z)$. The nature of such a degeneration of solution on parameter A requires additional study.

III. HELICITY OPERATOR

Let us solve a subsidiary task: find consequences of diagonalization of the helicity operator for electromagnetic spinor (related to plane waves)

$$\Sigma = -\frac{i}{2}\,[\partial_1(\sigma_1\otimes I + I\otimes\sigma_1) + \partial_2(\sigma_2\otimes I + I\otimes\sigma_2) + \partial_3(\sigma_3\otimes I + I\otimes\sigma_3)] = -\frac{i}{2}\,[\,\partial_1\Sigma_1 + \partial_2\Sigma_2 + \partial_3\Sigma_3\,]\,. \qquad (45)$$

Taking in mind the substitution for ξ:

$$\xi = e^{-i\omega t}e^{ik_1x}e^{ik_2y}e^{ik_3z}\left|\begin{array}{cc} f & h \\ h & g \end{array}\right|,$$

we have the following expression for Σ:

$$\Sigma = \frac{1}{2}[k_1(\sigma_1 \xi + \xi \tilde{\sigma}_1) + (\sigma_2 \xi + \xi \tilde{\sigma}_2) + k_2 + k_3(\sigma_3 \xi + \xi \tilde{\sigma}_3)] =$$

$$= \frac{1}{2}\left\{ k_1 \left| \begin{array}{cc} 2h & f+g \\ f+g & 2h \end{array} \right| + k_2 \left| \begin{array}{cc} -2ih & i(f-g) \\ i(f-g) & 2ih \end{array} \right| + k_3 \left| \begin{array}{cc} 2f & 0 \\ 0 & -2g \end{array} \right| \right\}. \tag{46}$$

Therefore, from the eigenvalue equation $\Sigma\, \xi = \sigma\, \xi$ we get four equations

$$\begin{array}{ll} 11 & (k_1 - ik_2)\; h + (k_3 - \sigma)\; f = 0\,, \\ 22 & (k_1 + ik_2)\; h - (k_3 + \sigma)\; g = 0\,, \\ 12 & \frac{1}{2}(k_1 + ik_2)\; f + \frac{1}{2}(k_1 - ik_2)\; g - \sigma\; h = 0\,, \\ 21 & \frac{1}{2}(k_1 + ik_2)\; f + \frac{1}{2}(k_1 - ik_2)\; g - \sigma\; h = 0\,; \end{array} \tag{47}$$

two last equations are the same. In (46), we have the system of three equations

$$\left| \begin{array}{ccc} (k_3 - \sigma) & 0 & (k_1 - ik_2) \\ 0 & -(k_3 + \sigma) & (k_1 + ik_2) \\ \frac{1}{2}(k_1 + ik_2) & \frac{1}{2}(k_1 - ik_2) & -\sigma \end{array} \right| \left| \begin{array}{c} f \\ g \\ h \end{array} \right| = 0\,. \tag{48}$$

For eigenvalues σ we have a cubic equation

$$(k_3 - \sigma)(k_3 + \sigma)\sigma + \frac{1}{2}(k_1 + ik_2)(k_1 - ik_2)(k_3 + \sigma) - \frac{1}{2}(k_1 + ik_2)(k_1 - ik_2)(k_3 - \sigma) = 0$$

or $\sigma(k_1^2 + k_2^2 + k_3^2 - \sigma^2) = 0$; so we obtain three roots

$$\sigma = 0, +k, -k; \quad k = \sqrt{k_1^2 + k_2^2 + k_3^2}\,. \tag{49}$$

Let us find solutions of the system (48) at $\sigma = 0$:

$$\left| \begin{array}{ccc} k_3 & 0 & (k_1 - ik_2) \\ 0 & -k_3 & (k_1 + ik_2) \\ \frac{1}{2}(k_1 + ik_2) & \frac{1}{2}(k_1 - ik_2) & 0 \end{array} \right| \left| \begin{array}{c} f \\ g \\ h \end{array} \right| = 0\,,$$

or

$$k_3 f + (k_1 - ik_2)h = 0\,, \quad -k_3 g + (k_1 + ik_2)h = 0,\,, \quad (k_1 + ik_2)f + (k_1 - ik_2)g = 0\,;$$

with two first equations the third one turns to be an identity $0 \equiv 0$; therefore we get

$$\sigma = 0, \qquad f = -\frac{k_1 - ik_2}{k_3} h\,, \quad g = +\frac{k_1 + ik_2}{k_3} h\,. \tag{50}$$

Let us find solutions of the system (48) at $\sigma = -k0$:

$$\left| \begin{array}{ccc} (k_3 + k) & 0 & (k_1 - ik_2) \\ 0 & -(k_3 - k) & (k_1 + ik_2) \\ \frac{1}{2}(k_1 + ik_2) & \frac{1}{2}(k_1 - ik_2) & k \end{array} \right| \left| \begin{array}{c} f \\ g \\ h \end{array} \right| = 0\,,$$

that is

$$(k_3 + k)f + (k_1 - ik_2)h = 0\,, \quad -(k_3 - k)g + (k_1 + ik_2)h = 0\,, \quad \frac{1}{2}(k_1 + ik_2)f + \frac{1}{2}(k_1 - ik_2)g + kh = 0\,;$$

with two first equations the third one turns to be an identity $0 \equiv 0$; therefore we get

$$\sigma = -k, \qquad f = -\frac{k_1 - ik_2}{k_3 + k} h, \quad g = +\frac{k_1 + ik_2}{k_3 - k} h\,. \tag{51}$$

Let us find solutions of the system (48) at $\sigma = +k$:

$$\begin{vmatrix} (k_3 - k) & 0 & (k_1 - ik_2) \\ 0 & -(k_3 + k) & (k_1 + ik_2) \\ \frac{1}{2}(k_1 + ik_2) & \frac{1}{2}(k_1 - ik_2) & -k \end{vmatrix} \begin{vmatrix} f \\ g \\ h \end{vmatrix} = 0 \,,$$

or

$$(k_3 - k)f + (k_1 - ik_2)h = 0 \,, \quad -(k_3 + k)g + (k_1 + ik_2)h = 0 \,, \quad \frac{1}{2}(k_1 + ik_2)f + \frac{1}{2}(k_1 - ik_2)g - kh = 0 \,;$$

with two first equations the third one becomes an identity, so we arrive at

$$\sigma = +k, \qquad f = -\frac{k_1 - ik_2}{k_3 - k}h, \quad g = +\frac{k_1 + ik_2}{k_3 + k}h \,. \tag{52}$$

Now we are to compare (50)–(52) with solutions of the Maxwell equations in spinor form

$$\begin{cases} (k_3 - \omega)f + (k_1 - ik_2)h = 0 \,, \\ (k_1 + ik_2)f - (k_3 + \omega)h = 0 \,, \end{cases} \quad \omega = +k, \quad f = -\frac{k_1 - ik_2}{k_3 - k}h \,;$$
$$\begin{cases} (k_1 - ik_2)g + (k_3 - \omega)h = 0 \,, \\ -(k_3 + \omega)g + (k_1 + ik_2)h = 0 \,, \end{cases} \quad \omega = +k, \quad g = +\frac{k_1 + ik_2}{k_3 + k}h \,. \tag{53}$$

We conclude that solution of the spinor Maxwell equations is the eigenstate with $\sigma = +1$

Now, let us transform helicity operator Σ (see (45)) to cylindric parabolic coordinates and tetrad. It is known (see in [...]) that 2-dimensional spinor ψ_0 in Cartesian tetrad can be related to 2-spinor ψ in cylindric parabolic tetrad by means of the following local gauge transformation:

$$\psi = s\psi_0, \quad s = \frac{1}{(u^2 + v^2)^{1/4}} \begin{vmatrix} \sqrt{u + iv} & 0 \\ 0 & \sqrt{u - iv} \end{vmatrix}, \; s^{-1} = s^{+} = \frac{1}{(u^2 + v^2)^{1/4}} \begin{vmatrix} \sqrt{u - iv} & 0 \\ 0 & \sqrt{u + iv} \end{vmatrix}. \tag{54}$$

Correspondingly, 2-rank electromagnetic spinors in two tetrads may be related by the following gauge transformation

$$\xi = S\xi_0 = (s \otimes s)\xi_0 = s\xi_0\tilde{s} = s\xi_0 s \,, \quad \xi_0 = S^{-1}\xi = (s^{-1} \otimes s^{-1})\xi = s^{-1}\xi s^{-1}. \tag{55}$$

From the eigenstate equation in Cartesian basis $-\frac{i}{2}[\partial_1\Sigma_1 + \partial_2\Sigma_2 + \partial_z\Sigma_3]\xi_0 = \sigma\xi_0$ it follows

$$-\frac{i}{2}\left[S\Sigma_1 S^{-1}(\frac{\partial}{\partial x^1} + S\frac{\partial S^{-1}}{\partial x^1}) + S\Sigma_2 S^{-1}(\frac{\partial}{\partial x^2} + S\frac{\partial S^{-1}}{\partial x^2}) + S\Sigma_3 S^{-1}(\frac{\partial}{\partial x^2} + S\frac{\partial S^{-1}}{\partial x^3})\right] = \sigma\xi \,, \tag{56}$$

recall that the matrix S depends only on (x^1, x^2) (or (u, v)). There exist identities

$$\begin{aligned} S\Sigma_1 S^{-1} &= (s \otimes s)[\sigma_1 \otimes I + I \otimes \sigma_1](s^{-1} \otimes s^{-1}) = s\sigma_1 s^{-1} \otimes I + I \otimes s\sigma_1 s^{-1} \,, \\ S\Sigma_2 S^{-1} &= (s \otimes s)[\sigma_2 \otimes I + I \otimes \sigma_2](s^{-1} \otimes s^{-1}) = s\sigma_2 s^{-1} \otimes I + I \otimes s\sigma_2 s^{-1} \,, \\ S\Sigma_3 S^{-1} &= (s \otimes s)[\sigma_3 \otimes I + I \otimes \sigma_3](s^{-1} \otimes s^{-1}) = s\sigma_3 s^{-1} \otimes I + I \otimes s\sigma_3 s^{-1} \,, \end{aligned} \tag{57}$$

and also identities

$$\begin{aligned} S\frac{\partial S^{-1}}{\partial x^1} &= (s \otimes s)\frac{\partial(s^{-1} \otimes s^{-1})}{\partial x^1} = s\frac{\partial s^{-1}}{\partial x^1} \otimes I + I \otimes s\frac{\partial s^{-1}}{\partial x^1} \,, \\ S\frac{\partial S^{-1}}{\partial x^2} &= (s \otimes s)\frac{\partial(s^{-1} \otimes s^{-1})}{\partial x^2} = s\frac{\partial s^{-1}}{\partial x^2} \otimes I + I \otimes s\frac{\partial s^{-1}}{\partial x^2} \,, \\ S\frac{\partial S^{-1}}{\partial x^3} &= (s \otimes s)\frac{\partial(s^{-1} \otimes s^{-1})}{\partial x^3} \equiv 0 \,. \end{aligned} \tag{58}$$

Taking these in mind, we obtain (see (56))

$$S\Sigma_1 S^{-1}(\frac{\partial}{\partial x^1} + S\frac{\partial S^{-1}}{\partial x^1}) = \left(s\sigma_1 s^{-1} \otimes I + I \otimes s\sigma_1 s^{-1}\right)\left(\frac{\partial}{\partial x^1} + s\frac{\partial s^{-1}}{\partial x^1} \otimes I + I \otimes s\frac{\partial s^{-1}}{\partial x^1}\right),$$

that is

$$S\Sigma_1 S^{-1}(\frac{\partial}{\partial x^1} + S\frac{\partial S^{-1}}{\partial x^1}) = (s\sigma_1 s^{-1} \otimes I + I \otimes s\sigma_1 s^{-1})\ \frac{\partial}{\partial x^1} +$$
$$+ s\sigma_1 \frac{\partial s^{-1}}{\partial x^1} \otimes I + I \otimes s\sigma_1 \frac{\partial s^{-1}}{\partial x^1} + s\sigma_1 s^{-1} \otimes s\frac{\partial s^{-1}}{\partial x^1} + s\frac{\partial s^{-1}}{\partial x^1} \otimes s\sigma_1 s^{-1}\,. \tag{59}$$

Similarly, derive

$$S\Sigma_2 S^{-1}(\frac{\partial}{\partial x^2} + S\frac{\partial S^{-1}}{\partial x^2}) = (s\sigma_2 s^{-1} \otimes I + I \otimes s\sigma_2 s^{-1})\ \frac{\partial}{\partial x^2} +$$
$$+ s\sigma_2 \frac{\partial s^{-1}}{\partial x^2} \otimes I + I \otimes s\sigma_2 \frac{\partial s^{-1}}{\partial x^2} + s\sigma_2 s^{-1} \otimes s\frac{\partial s^{-1}}{\partial x^2} + s\frac{\partial s^{-1}}{\partial x^2} \otimes s\sigma_2 s^{-1}\,, \tag{60}$$

and

$$S\Sigma_3 S^{-1}(\frac{\partial}{\partial x^3} + S\frac{\partial S^{-1}}{\partial x^3}) = S\Sigma_3 S^{-1}\frac{\partial}{\partial x^3} = (s\sigma_3 s^{-1} \otimes I + I \otimes s\sigma_3 s_3)\frac{\partial}{\partial x^3}\,. \tag{61}$$

Let us write down a general structure for operator Σ:

$$\Sigma = -\frac{i}{2}\left\{\left(s\sigma_1 s^{-1} \otimes I + I \otimes s\sigma_1 s^{-1}\right)\ \frac{\partial}{\partial x^1} + \left(s\sigma_2 s^{-1} \otimes I + I \otimes s\sigma_2 s^{-1}\right)\ \frac{\partial}{\partial x^2} + \left(s\sigma_3 s^{-1} \otimes I + I \otimes s\sigma_3 s_3\right)\ \frac{\partial}{\partial x^3} + \right.$$
$$+ \left(s\sigma_1 s^{-1} \cdot s\frac{\partial s^{-1}}{\partial x^1} \otimes I + I \otimes s\sigma_1 s^{-1} \cdot s\frac{\partial s^{-1}}{\partial x^1}\right) + \left(s\sigma_2 s^{-1} \cdot s\frac{\partial s^{-1}}{\partial x^2} \otimes I + s\sigma_2 s^{-1} \cdot s\frac{\partial s^{-1}}{\partial x^2} \otimes I\right) +$$
$$\left. + \left(s\sigma_1 s^{-1} \otimes s\frac{\partial s^{-1}}{\partial x^1} + s\frac{\partial s^{-1}}{\partial x^1} \otimes s\sigma_1 s^{-1}\right) + \left(s\sigma_2 s^{-1} \otimes s\frac{\partial s^{-1}}{\partial x^2} + s\frac{\partial s^{-1}}{\partial x^2} \otimes s\sigma_2 s^{-1}\right)\right\}. \tag{62}$$

Readily find the formulas

$$s\sigma_3 s^{-1} = \sigma_3\,,$$
$$s\sigma_1 s^{-1} = \frac{1}{\sqrt{u^2+v^2}}\begin{vmatrix} 0 & u+iv \\ u-iv & 0 \end{vmatrix} = \frac{1}{\sqrt{u^2+v^2}}(u\sigma_1 - v\sigma_2)\,, \tag{63}$$
$$s\sigma_2 s^{-1} = \frac{1}{\sqrt{u^2+v^2}}\begin{vmatrix} 0 & v-iu \\ v+iu & 0 \end{vmatrix} = \frac{1}{\sqrt{u^2+v^2}}(v\sigma_1 + u\sigma_2)\,.$$

Taking in mind relations

$$\frac{\partial u}{\partial x^1} = \frac{u}{u^2+v^2}\,, \quad \frac{\partial v}{\partial x^1} = \frac{-v}{u^2+v^2}\,, \quad \frac{\partial u}{\partial x^2} = \frac{v}{u^2+v^2}\,, \quad \frac{\partial v}{\partial x^2} = \frac{u}{u^2+v^2}\,,$$

derive the formulas

$$\frac{\partial}{\partial x^1} = \frac{1}{u^2+v^2}(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v})\,, \quad \frac{\partial}{\partial x^2} = \frac{1}{u^2+v^2}(v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v})\,. \tag{64}$$

Let us consider three first terms in (62):

$$(s\sigma_3 s^{-1} \otimes I + I \otimes s\sigma_3 s_3)\frac{\partial}{\partial x^3} + [s\sigma_1 s^{-1} \otimes I + I \otimes s\sigma_1 s^{-1}]\ \frac{\partial}{\partial x^1} + [s\sigma_2 s^{-1} \otimes I + I \otimes s\sigma_2 s^{-1}]\ \frac{\partial}{\partial x^2} =$$
$$= (\sigma_3 \otimes + I \otimes \sigma_3)\frac{\partial}{\partial x^3} + \frac{1}{\sqrt{u^2+v^2}}\frac{1}{u^2+v^2} \times$$
$$\times \left\{[(u\sigma_1 - v\sigma_2) \otimes I + I \otimes (u\sigma_1 - v\sigma_2)](u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}) + [(v\sigma_1 + u\sigma_2) \otimes I + I \otimes (v\sigma_1 + u\sigma_2)](v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v})\right\} =$$
$$= (\sigma_3 \otimes + I \otimes \sigma_3)\frac{\partial}{\partial x^3} + \frac{1}{\sqrt{u^2+v^2}}\left[(\sigma_1\frac{\partial}{\partial u} + \sigma_2\frac{\partial}{\partial v}) \otimes I + I \otimes (\sigma_1\frac{\partial}{\partial u} + \sigma_2\frac{\partial}{\partial v})\right]\,. \tag{65}$$

Further, consider the term

$$s\frac{\partial s^{-1}}{\partial x^1} = \frac{1}{u^2+v^2} s\left\{u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right\} s^{-1}\frac{1}{u^2+v^2}\frac{1}{(u^2+v^2)^{1/4}}\begin{vmatrix} \sqrt{u+iv} & 0 \\ 0 & \sqrt{u-iv} \end{vmatrix} \times$$

$$\times \Bigg\{ -\frac{u^2}{2(u^2+v^2)(u^2+v^2)^{1/4}} \begin{vmatrix} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{vmatrix} + \frac{1}{2(u^2+v^2)^{1/4}} \begin{vmatrix} \frac{u}{\sqrt{u-iv}} & 0 \\ 0 & \frac{u}{\sqrt{u+iv}} \end{vmatrix} +$$

$$+\frac{v^2}{2(u^2+v^2)(u^2+v^2)^{1/4}} \begin{vmatrix} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{vmatrix} + \frac{1}{2(u^2+v^2)^{1/4}} \begin{vmatrix} \frac{iv}{\sqrt{u-iv}} & 0 \\ 0 & \frac{-iv}{\sqrt{u+iv}} \end{vmatrix} \Bigg\} =$$

$$= \frac{1}{2} \frac{1}{u^2+v^2} \frac{1}{(u^2+v^2)^{1/2}} \begin{vmatrix} \sqrt{u+iv} & 0 \\ 0 & \sqrt{u-iv} \end{vmatrix} \times$$

$$\times \Bigg\{ -\frac{u^2}{(u^2+v^2)} \begin{vmatrix} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{vmatrix} + \begin{vmatrix} \frac{u}{\sqrt{u-iv}} & 0 \\ 0 & \frac{u}{\sqrt{u+iv}} \end{vmatrix} +$$

$$+\frac{v^2}{(u^2+v^2)} \begin{vmatrix} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{vmatrix} + \begin{vmatrix} \frac{iv}{\sqrt{u-iv}} & 0 \\ 0 & \frac{-iv}{\sqrt{u+iv}} \end{vmatrix} \Bigg\} =$$

$$= \frac{1}{2} \frac{v^2-u^2}{(u^2+v^2)^2} + \frac{1}{2} \frac{\sqrt{u+iv}\sqrt{u-iv}}{(u^2+v^2)^2} \begin{vmatrix} \frac{(u+iv)\sqrt{u+iv}}{\sqrt{u-iv}} & 0 \\ 0 & \frac{(u-iv)\sqrt{u-iv}}{\sqrt{u+iv}} \end{vmatrix} =$$

$$= \frac{1}{2} \frac{v^2-u^2}{(u^2+v^2)^2} + \frac{1}{2} \frac{1}{(u^2+v^2)^2} \begin{vmatrix} (u+iv)^2 & 0 \\ 0 & (u-iv)^2 \end{vmatrix} ,$$

so, arriving at a simple result

$$s\frac{\partial s^{-1}}{\partial x^1} = \frac{2i\,uv}{2(u^2+v^2)^2} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \frac{2i\,uv}{2(u^2+v^2)^2}\,\sigma_3\,. \qquad (66)$$

Similarly, consider the second term

$$s\frac{\partial s^{-1}}{\partial x^2} = \frac{1}{u^2+v^2} s \left\{ v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} \right\} s^{-1} = \frac{1}{u^2+v^2} \frac{1}{(u^2+v^2)^{1/4}} \begin{vmatrix} \sqrt{u+iv} & 0 \\ 0 & \sqrt{u-iv} \end{vmatrix} \times$$

$$\times \Bigg\{ -\frac{uv}{2(u^2+v^2)(u^2+v^2)^{1/4}} \begin{vmatrix} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{vmatrix} + \frac{1}{2(u^2+v^2)^{1/4}} \begin{vmatrix} \frac{v}{\sqrt{u-iv}} & 0 \\ 0 & \frac{v}{\sqrt{u+iv}} \end{vmatrix} -$$

$$-\frac{uv}{2(u^2+v^2)(u^2+v^2)^{1/4}} \begin{vmatrix} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{vmatrix} + \frac{1}{2(u^2+v^2)^{1/4}} \begin{vmatrix} \frac{-iu}{\sqrt{u-iv}} & 0 \\ 0 & \frac{+iu}{\sqrt{u+iv}} \end{vmatrix} \Bigg\} =$$

$$= \frac{1}{2} \frac{1}{u^2+v^2} \frac{1}{(u^2+v^2)^{1/2}} \begin{vmatrix} \sqrt{u+iv} & 0 \\ 0 & \sqrt{u-iv} \end{vmatrix} \times$$

$$\times \Bigg\{ -\frac{uv}{(u^2+v^2)} \begin{vmatrix} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{vmatrix} + \begin{vmatrix} \frac{v}{\sqrt{u-iv}} & 0 \\ 0 & \frac{v}{\sqrt{u+iv}} \end{vmatrix} -$$

$$-\frac{uv}{(u^2+v^2)} \begin{vmatrix} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{vmatrix} + \begin{vmatrix} \frac{-iu}{\sqrt{u-iv}} & 0 \\ 0 & \frac{+iu}{\sqrt{u+iv}} \end{vmatrix} \Bigg\} =$$

$$= -\frac{1}{2}\frac{2uv}{(u^2+v^2)^2} + \frac{1}{2}\frac{\sqrt{u+iv}\sqrt{u-iv}}{(u^2+v^2)^2}\left|\begin{array}{cc} \frac{(v-iu)\sqrt{u+iv}}{\sqrt{u-iv}} & 0 \\ 0 & \frac{(v+iu)\sqrt{u-iv}}{\sqrt{u+iv}} \end{array}\right| =$$

$$= -\frac{1}{2}\frac{2uv}{(u^2+v^2)^2} + \frac{1}{2}\frac{1}{(u^2+v^2)^2}\left|\begin{array}{cc} (u+iv)(v-iu) & 0 \\ 0 & (u-iv)(v+iu) \end{array}\right| ,$$

so that

$$s\frac{\partial s^{-1}}{\partial x^2} = \frac{i\,(v^2-u^2)}{2(u^2+v^2)^2}\left|\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right| = \frac{i\,(v^2-u^2)}{2(u^2+v^2)^2}\,\sigma_3\,. \tag{67}$$

Taking into account the formulas (63)–(67, we transform expression for Σ in (62) to the following form

$$\Sigma = -\frac{i}{2}\Big\{(\sigma_3 \otimes +I\otimes\sigma_3)\frac{\partial}{\partial x^3} + \frac{1}{\sqrt{u^2+v^2}}\Big[(\sigma_1\frac{\partial}{\partial u} + \sigma_2\frac{\partial}{\partial v})\otimes I + I\otimes(\sigma_1\frac{\partial}{\partial u} + \sigma_2\frac{\partial}{\partial v})\Big] +$$
$$+\frac{1}{\sqrt{u^2+v^2}}\frac{2i\,uv}{2(u^2+v^2)^2}\;[(u\sigma_1 - v\sigma_2)\sigma_3\otimes I + I\otimes(u\sigma_1-v\sigma_2)\sigma_3] +$$
$$+\frac{1}{\sqrt{u^2+v^2}}\frac{i\,(v^2-u^2)}{2(u^2+v^2)^2}\,[(v\sigma_1+u\sigma_2)\;\sigma_3\times I + I\otimes(v\sigma_1+u\sigma_2)\;\sigma_3]$$
$$+\frac{1}{\sqrt{u^2+v^2}}\frac{2i\,uv}{2(u^2+v^2)^2}\,[(u\sigma_1-v\sigma_2)\otimes\sigma_3 + \sigma_3\otimes(u\sigma_1-v\sigma_2)] +$$
$$+\frac{1}{\sqrt{u^2+v^2}}\frac{i\,(v^2-u^2)}{2(u^2+v^2)^2}\,[(v\sigma_1+u\sigma_2)\otimes\;\sigma_3 + \sigma_3\otimes(v\sigma_1+u\sigma_2)]\Big\}\,.$$

It may be re-written differently

$$\Sigma = -\frac{i}{2}\Big\{(\sigma_3\otimes +I\otimes\sigma_3)\frac{\partial}{\partial x^3} + \frac{1}{\sqrt{u^2+v^2}}\Big\{(\sigma_1\partial_u + \sigma_2\frac{\partial}{\partial v})\otimes I + I\otimes(\sigma_1\partial_u + \sigma_2\frac{\partial}{\partial v}) +$$
$$+\frac{2uv}{2(u^2+v^2)^2}\;[(v\sigma_1+u\sigma_2)\otimes I + I\otimes(v\sigma_1+u\sigma_2) + (u\sigma_1-v\sigma_2)\otimes i\sigma_3 + i\sigma_3\otimes(u\sigma_1-v\sigma_2)] +$$
$$+\frac{v^2-u^2}{2(u^2+v^2)^2}[(-u\sigma_1+v\sigma_2)\otimes I + I\otimes(-u\sigma_1+v\sigma_2) + \;+(v\sigma_1+u\sigma_2)\otimes i\sigma_3 + i\sigma_3\otimes(v\sigma_1+u\sigma_2)]\;\}\;\}\,. \tag{68}$$

The eigenvalue equation $\Sigma\xi = \sigma\xi$ leads to

$$\{(\sigma_1\partial_u+\sigma_2\partial_v)\otimes I + I\otimes(\sigma_1\partial_u+\sigma_2\partial_v) + \frac{2uv}{2(u^2+v^2)^2}\;[(v\sigma_1+u\sigma_2)\otimes I + I\otimes(v\sigma_1+u\sigma_2)] +$$
$$+\frac{v^2-u^2}{2(u^2+v^2)^2}[(-u\sigma_1+v\sigma_2)\otimes I + I\otimes(-u\sigma_1+v\sigma_2)] + \frac{2uv}{2(u^2+v^2)^2}\;[(u\sigma_1-v\sigma_2)\otimes i\sigma_3 + i\sigma_3\otimes(u\sigma_1-v\sigma_2)] +$$
$$+\frac{v^2-u^2}{2(u^2+v^2)^2}[(v\sigma_1+u\sigma_2)\otimes i\sigma_3 + i\sigma_3\otimes(v\sigma_1+u\sigma_2)]\Big\}\,\xi = \sqrt{u^2+v^2}\;(2i\sigma - ik\sigma_3\otimes -ikI\otimes\sigma_3)\;\xi\,. \tag{69}$$

Regrouping the terms within the lines 2-3, and also within the lines 3-4, we obtain more simple form for the equation:

$$[(\sigma_1\partial_u+\sigma_2\partial_v)\otimes I + I\otimes(\sigma_1\partial_u+\sigma_2\partial_v)]\xi + \frac{1}{2(u^2+v^2)}\times$$
$$\times\{(u\sigma_1+v\sigma_2)\otimes I + I\otimes(u\sigma_1+v\sigma_2) + (v\sigma_1-u\sigma_2)\otimes i\sigma_3 + i\sigma_3\otimes(v\sigma_1-u\sigma_2)\}\,\xi =$$
$$= \sqrt{u^2+v^2}\;(2i\sigma - ik\sigma_3\otimes I - ikI\otimes\sigma_3)\;\xi, \quad \text{where} \quad h = \left|\begin{array}{cc} f & h \\ h & g \end{array}\right|. \tag{70}$$

First we calculate

$$(\sigma_1\partial_u+\sigma_2\partial_v)\otimes I\left|\begin{array}{cc} f & h \\ h & g \end{array}\right| = \left|\begin{array}{cc} 0 & \partial_u - i\partial_v \\ \partial_u + i\partial_v & 0 \end{array}\right|\left|\begin{array}{cc} f & h \\ h & g \end{array}\right| = \left|\begin{array}{cc} (\partial_u - i\partial_v)h & (\partial_u - i\partial_v)g \\ (\partial_u + i\partial_v)f & (\partial_u + i\partial_v)h \end{array}\right|;$$

and

$$I \otimes (\sigma_1 \partial_u + \sigma_2 \frac{\partial}{\partial v}) \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| = \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| \left| \begin{array}{cc} 0 & \partial_u + i\partial_v \\ \partial_u - i\partial_v & 0 \end{array} \right| = \left| \begin{array}{cc} (\partial_u - i\partial_v)h & (\partial_u + i\partial_v)f \\ (\partial_u - i\partial_v)g & (\partial_u + i\partial_v)h \end{array} \right| ;$$

Their sum equals to

$$(\sigma_1 \partial_u + \sigma_2 \frac{\partial}{\partial v}) \otimes I \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| + I \otimes (\sigma_1 \partial_u + \sigma_2 \frac{\partial}{\partial v}) \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| =$$
$$= \left| \begin{array}{cc} 2(\partial_u - i\partial_v)h & (\partial_u + i\partial_v)f + (\partial_u - i\partial_v)g \\ (\partial_u + i\partial_v)f + (\partial_u - i\partial_v)g & 2(\partial_u + i\partial_v)h \end{array} \right| . \tag{71}$$

Then we calculates the terms

$$[(u\sigma_1 + v\sigma_2) \otimes I + I \otimes (u\sigma_1 + v\sigma_2)]\xi = \left| \begin{array}{cc} 0 & u - iv \\ u + iv & 0 \end{array} \right| \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| + \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| \left| \begin{array}{cc} 0 & u + iv \\ u - iv & 0 \end{array} \right| =$$

$$= \left| \begin{array}{cc} (u - iv)h & (u - iv)g \\ (u + iv)f & (u + iv)h \end{array} \right| + \left| \begin{array}{cc} (u - iv))h & (u + iv)f \\ (u - iv)g & (u + iv)h \end{array} \right| = \left| \begin{array}{cc} 2(u - iv)h & (u + iv)f + (u - iv)g \\ (u + iv)f + (u - iv)g & 2(u + iv)h \end{array} \right| ,$$

and

$$[(v\sigma_1 - u\sigma_2) \otimes i\sigma_3 + i\sigma_3 \otimes (v\sigma_1 - u\sigma_2)]\xi =$$

$$= \left| \begin{array}{cc} 0 & v + iu \\ v - iu & 0 \end{array} \right| \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| \left| \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right| + \left| \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right| \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| \left| \begin{array}{cc} 0 & v - iu \\ v + iu & 0 \end{array} \right| =$$

$$= \left| \begin{array}{cc} (iv - u)h & (-iv + u)g \\ (iv + u)f & (-iv - u)h \end{array} \right| + \left| \begin{array}{cc} (iv - u)h & (iv + u)f \\ (-iv + u)g & (-iv - u)h \end{array} \right| = \left| \begin{array}{cc} 2(iv - u)h & (iv + u)f + (-iv + u)g \\ (iv + u)f + (-iv + u)g & 2(-iv - u)h \end{array} \right| ;$$

their sum equals to

$$... = 2 \left| \begin{array}{cc} 0 & (u + iv)f + (u - iv)g \\ (u + iv)f + (u - iv)g & 0 \end{array} \right| . \tag{72}$$

Besides, we find

$$(2i\sigma - ik\sigma_3 \otimes -ikI \otimes \sigma_3)\, \xi = 2i \left| \begin{array}{cc} (\sigma - k)f & \sigma h \\ \sigma h & (\sigma + k)g \end{array} \right| . \tag{73}$$

Taking into account relations (71)–(73), we reduce eq. to the form

$$\left| \begin{array}{cc} 2(\partial_u - i\partial_v)h & (\partial_u + i\partial_v)f + (\partial_u - i\partial_v)g \\ (\partial_u + i\partial_v)f + (\partial_u - i\partial_v)g & 2(\partial_u + i\partial_v)h \end{array} \right| +$$
$$+ \frac{1}{(u^2 + v^2)^2} \left| \begin{array}{cc} 0 & (u + iv)f + (u - iv)g \\ (u + iv)f + (u - iv)g & 0 \end{array} \right| = 2i\sqrt{u^2 + v^2} \left| \begin{array}{cc} (\sigma - k)f & \sigma h \\ \sigma h & (\sigma + k)g \end{array} \right| . \tag{74}$$

Whence we derive four equations. First, let us consider the following ones:

$$11,\ 2(\partial_u - i\partial_v)h = 2i\sqrt{u^2 + v^2}(\sigma - k)f\,, \qquad 22,\ 2(\partial_u + i\partial_v)h = 2i\sqrt{u^2 + v^2}(\sigma + k)g\,,$$

or differently

$$(\partial_u - i\partial_v)h = i(\sigma - k)\bar{f}\,, \quad (\partial_u + i\partial_v)h = i(\sigma + k)\bar{g}\,; \tag{75}$$

recall the notations $\bar{f} = \sqrt{u^2+v^2} f$, $\bar{g} = \sqrt{u^2+v^2} g$. Let us introduce new variables $\bar{f}+\bar{g} = F$, $\bar{f}-\bar{g} = G$, then summing and subtracting equations in (75) we obtain the system of two linear equations with respect to F and G:

$$\sigma F - kG = -2i\partial_u h \qquad -kF + \sigma G = -2\partial_v h \; .$$

Its solution is

$$F = \frac{2}{\sigma^2 - k^2} [-i\sigma\partial_u - k\partial_v]\, h \,, \qquad G = \frac{2}{\sigma^2 - k^2} [-ik\partial_u - \sigma\partial_v]\, h \,. \tag{76}$$

These formulas may be compared with consequences of the Maxwell equations (29):

$$F = \frac{2}{\omega^2 - k^2} [-i\omega\partial_u - k\partial_v]\, h \,, \quad G = \frac{2}{\omega^2 - k^2} [-ik\partial_u - \omega\partial_v]\, h \,. \tag{77}$$

Relations (76) and (77) coincide when identifying σ and ω. N ow consider two remaining equations:

$$\begin{array}{ll} 12, & (\partial_u + i\partial_v) f + (\partial_u - i\partial_v) g + \dfrac{1}{(u^2+v^2)} [(u+iv)f + (u-iv)g] = 2i\sqrt{u^2+v^2}\sigma h \,, \\ 21, & (\partial_u + i\partial_v) f + (\partial_u - i\partial_v) g + \dfrac{1}{(u^2+v^2)} [(u+iv)f + (u-iv)g] = 2i\sqrt{u^2+v^2}\sigma h \,; \end{array} \tag{78}$$

the coincide with each other. With taking in mind identities of the type

$$\partial_u f = \partial_u \frac{\bar{f}}{\sqrt{u^2+v^2}} = \frac{1}{\sqrt{u^2+v^2}} (\partial_u - \frac{u}{u^2+v^2}) \bar{f} \,,$$

equation from (78) takes the simpler form

$$\frac{1}{\sqrt{u^2+v^2}} (\partial_u - \frac{u}{u^2+v^2}) F + \frac{i}{\sqrt{u^2+v^2}} (\partial_v - \frac{v}{u^2+v^2}) G \frac{1}{(u^2+v^2)} \frac{1}{\sqrt{u^2+v^2}} [uF + ivG] = 2i\sqrt{u^2+v^2}\sigma h \,,$$

which after regrouping the terms leads to

$$\frac{1}{u^2+v^2} (\partial_u - \frac{u}{u^2+v^2}) F + \frac{i}{u^2+v^2} (\partial_v - \frac{v}{u^2+v^2}) G + \frac{1}{(u^2+v^2)^2} [uF + ivG] = 2i\sigma h \,.$$

Its final form is

$$\frac{1}{u^2+v^2} \partial_u F + \frac{i}{u^2+v^2} \partial_v G = 2i\sigma h \,. \tag{79}$$

We may notice that the last equation (79) coincides with the third equation in the system (28)

$$-2i\omega h + \frac{1}{u^2+v^2} \partial_u F + \frac{i}{u^2+v^2} \partial_v G = 0 \,,$$

remembering on identity $\sigma = \omega$. Recall that the fourth equation in(28) is equivalent the third one.

Thus, we have shown that diagonalization of the helicity operator for 2=rank symmetric spinor it follows the system of equations coincided with that following from the Maxwell equation, when identifying the eigenvalue σ with the frequency ω. An this fact does not depend on the choice of coordinates and tetrad, Cartesian or cylindric parabolic.

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