Quasi-harmonic approximation in the case of Cornell potential

A.V. Baran* and V.V. Kudryashov†
Institute of Physics, National Academy of Sciences of Belarus
68 Nezavisimosti Ave., 220072, Minsk, Belarus

The recently proposed quasi-harmonic approximation is improved with the help of Langer's exponential substitution in order to solve the radial Schrödinger equation. This approach is applied in the case of confining Cornell potential, which describes quark-antiquark interaction. As a result the new approximate radial wave functions and energy levels are obtained for quarkonium bound states.

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I. INTRODUCTION

It is widely accepted that the quark-antiquark interaction is described by the confining Cornell potential [1]

$$V(r) = -\frac{A}{r} + Br. \tag{1}$$

The quantum mechanical non-relativistic consideration of quarkonium bound states leads to the radial Schrödinger equation

$$\frac{d^2\psi(r)}{dr^2} = \frac{2m}{\hbar^2}\left(V(r) + \frac{\hbar^2 l(l+1)}{2mr^2} - E\right)\psi(r). \tag{2}$$

where $l = 0, 1, 2 \ldots$. Many investigations have been devoted to solving Eq.(2) by means of different approximation procedures (see, e.g., [2–4] and references therein) based on perturbation theory, the variational method and their combinations. If we deal with perturbation theory or with variational method then similar question arise. How to find the unperturbed Hamiltonian or how to find the trial function for an arbitrary given potential? Universal answers are absent. In contrast, the WKB approximation is directly determined by a given potential. However, since the conventional WKB approximation has nonphysical singularities, this method cannot be used for adequate description of the physical systems. An old problem in semiclassical analysis is the development of global uniform approximations to the wave functions.

In [5], the satisfactory quasi-harmonic approximation for eigenfunction with the correct behavior at the turning points was constructed by means of the explicit summation of the asymptotic partial series, which were extracted from the complete WKB series. In [5], the proposed method was verified in the case of one-dimensional anharmonic oscillator with positive second derivative of potential. In [6], the quasi-harmonic approximation was generalized and applied to the modified Pöschl-Teller potential with inflection points. In the present paper, we use new approach in order to obtain approximate wave functions and energy levels for quarkonium bound states.

II. APPROXIMATE LOGARITHMIC DERIVATIVES

Introducing the dimensionless quantities

$$x = \left(\frac{2mB}{\hbar^2}\right)^{1/3} r, \quad e = \left(\frac{2m}{\hbar^2 B^2}\right)^{1/3} E, \quad k = \left(\frac{4m^2}{\hbar^4 B}\right)^{1/3} A \tag{3}$$

it is convenient to rewrite Eq.(2) in the form

$$\frac{d^2\psi(x)}{dx^2} = \left(-\frac{k}{x} + x + \frac{l(l+1)}{x^2} - e\right)\psi(x). \tag{4}$$

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*Electronic address: `a.baran@dragon.bas-net.by`
†Electronic address: `kudryash@dragon.bas-net.by`

It is known [7, 8] that a suitable transformation of the initial equation improves results of an approximation technique. We consider the Langer's exponential substitution [7]

$$x = \exp(q), \quad \psi(x) = x^{1/2}\Psi(q), \tag{5}$$

which provides the correct limiting behavior of the approximate wave functions at the origin in the WKB approach. The transformed equation is

$$\frac{d^2\Psi(q)}{dq^2} = Q(q)\Psi(q). \tag{6}$$

The function

$$Q(q) = -k\exp(q) + \exp(3q) - e\exp(2q) + (l + 1/2)^2 \tag{7}$$

has the inflection point

$$q_0 = \log\left(\frac{2e + \sqrt{4e^2 + 9k}}{9}\right) \tag{8}$$

and two turning points q_- and q_+, where $Q(q) = 0$.

The function $Q(q)$ is presented in Fig. 1 for $k = 1$, $l = 1$ and $e = 2.8$.

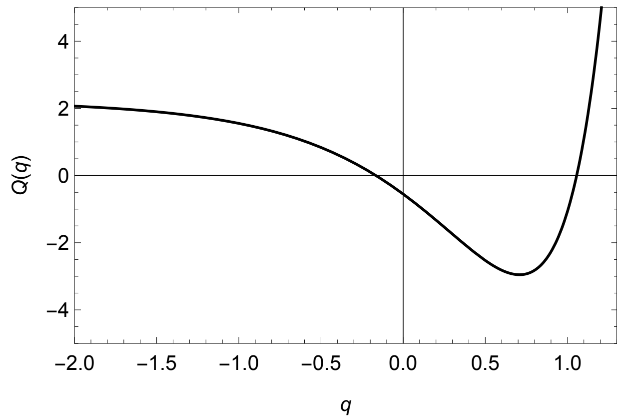

FIG. 1: The function $Q(q)$.

The quasi-harmonic approach deals with a logarithmic derivative

$$Y(q) = \frac{1}{\Psi(q)}\frac{d\Psi(q)}{dq}, \tag{9}$$

which satisfies the Riccati equation

$$\frac{dY(q)}{dq} + Y^2(q) = Q(q). \tag{10}$$

The approximate quasi-harmonic logarithmic derivative has the following structure

$$Y_{app}(q) = b(q)y(a(q), z(q)), \tag{11}$$

where

$$\begin{aligned} z(q) &= 2^{1/4}\frac{Q'(q)}{|Q''(q)|^{3/4}}, \quad b(q) = 2^{1/4}\frac{Q''(q)}{|Q''(q)|^{3/4}}, \\ a(q) &= \frac{1}{\sqrt{2|Q''(q)|}}\left(Q(q) - \frac{(Q'(q))^2}{2Q''(q)}\right). \end{aligned} \tag{12}$$

The functions $y(a,z)$ are solutions of the comparison equation, whose type is determined by sign of $Q''(q)$. Here we use notations $Q'(q) = dQ(q)/dq$ and $Q''(q) = d^2Q(q)/dq^2$.

If $Q''(q) > 0$ then the function $y(a,z)$ satisfies the basic Riccati equation

$$\frac{dy}{dz} + y^2 = \frac{z^2}{4} + a. \tag{13}$$

We get solutions

$$y_1^{\pm}(a,z) = \frac{1}{U(a,\pm z)} \frac{dU(a,\pm z)}{dz} \tag{14}$$

in the region, where $Q(q) > 0$, and we derive other solutions

$$\tilde{y}_1^{\pm}(a,z) = \frac{1}{U(-a,\pm iz)} \frac{dU(-a,\pm iz)}{dz}. \tag{15}$$

in the region, where $Q(q) < 0$. These solutions are expressed via the parabolic cylinder functions [9].

If $Q''(q) < 0$ then the function $y(a,z)$ satisfies the different basic Riccati equation

$$\frac{dy}{dz} + y^2 = -\frac{z^2}{4} + a. \tag{16}$$

Now we choose the following particular solutions

$$y_2^{\pm}(a,z) = \frac{1}{W(a,\mp z)} \frac{dW(a,\mp z)}{dz} \tag{17}$$

in the region, where $Q(q) > 0$, and we choose other particular solutions

$$\tilde{y}_2^{\pm}(a,z) = \frac{1}{U(\mp ia, -e^{\pm i\pi/4}z)} \frac{dU(\mp ia, -e^{\pm i\pi/4}z)}{dz} \tag{18}$$

in the region, where $Q(q) < 0$.

We select just solutions (18) in order to fulfill the smooth transition

$$b(q)\tilde{y}_2^{\pm}(a(q), z(q)) \to b(q)\tilde{y}_1^{\pm}(a(q), z(q)), \tag{19}$$

at the inflection points q_0.

Thus, the approximate logarithmic derivatives are given by formulas

$$Y_{app}^{\pm}(q) = \begin{cases} b(q)y_1^{\pm}(a(q), z(q)), & Q''(q) > 0, \\ b(q)y_2^{\pm}(a(q), z(q)), & Q''(q) < 0 \end{cases} \tag{20}$$

for $Q(q) > 0$, and

$$\tilde{Y}_{app}^{\pm}(q) = \begin{cases} b(q)\tilde{y}_1^{\pm}(a(q), z(q)), & Q''(q) > 0, \\ b(q)\tilde{y}_2^{\pm}(a(q), z(q)), & Q''(q) < 0 \end{cases} \tag{21}$$

for $Q(q) < 0$.

III. APPROXIMATE WAVE FUNCTIONS

In accordance with Eq.(5) the approximate solution $\psi_{app}(x)$ of the radial equation (4) is expressed via the approximate solution $\Psi_{app}(q)$ of Eq.(6) with the help of relation

$$\psi_{app}(x) = x^{1/2}\Psi_{app}(q). \tag{22}$$

Using the obtained approximate logarithmic derivatives (20) and (21) we can construct the approximate wave functions $\Psi_{app}(q)$. In the region $-\infty < q < q_-$, where $Q''(q) < 0$ and $Q(q) > 0$, we choose the approximate logarithmic derivative $Y^-_{app}(q)$ in order to provide the correct limiting behavior of the approximate wave function at $q \to -\infty$ ($x \to 0$). In the region $q_+ < q < \infty$, where $Q''(q) > 0$ and $Q(q) > 0$, we choose the approximate logarithmic derivative $Y^+_{app}(q)$ in order to provide the correct asymptotic behavior of the approximate wave function at $q \to \infty$ ($x \to \infty$). In the region between the turning points $q_- < q < q_+$, the oscillatory behavior of the approximate wave function is provided of we use the approximate logarithmic derivatives $Y^\pm_{app}(q)$.

The continuous at the turning points approximate function $\Psi_{app}(q)$ is presented by formulas

$$\Psi_{app}(q) = \begin{cases} N\Psi_1(q), & q < q_-, \\ N\Psi_2(q), & q_- < q < q_+, \\ N\Psi_3(q), & q > q_+, \end{cases} \tag{23}$$

where

$$\Psi_1(q) = \cos(\phi_-) \exp\left(- \int_q^{q_-} Y^-_{app}(q')\, dq' \right), \tag{24}$$

$$\Psi_2(q) = \exp\left(\int_{q_-}^{q} \frac{\tilde{Y}^-_{app}(q') + \tilde{Y}^+_{app}(q')}{2}\, dq' \right) \cos\left(\int_{q_-}^{q} \frac{\tilde{Y}^-_{app}(q') - \tilde{Y}^+_{app}(q')}{2i}\, dq' + \phi_- \right), \tag{25}$$

$$\Psi_3(q) = \Psi_2(q_+) \exp\left(\int_{q_+}^{q} Y^+_{app}(q')\, dq' \right). \tag{26}$$

The value of factor N is determined by the normalization condition $\langle \psi_{app} | \psi_{app} \rangle = 1$.

The continuity conditions at the turning points q_- and q_+ for the first derivative of $\Psi_{app}(q)$ lead to the new quantization rule

$$\int_{q_-}^{q_+} \frac{\tilde{Y}^-_{app}(q) - \tilde{Y}^+_{app}(q)}{2i}\, dq + \phi_- - \phi_+ = n\pi, \quad n = 0, 1, 2, \ldots, \tag{27}$$

where

$$\phi_\pm = \arctan\left(i \frac{\tilde{Y}^-_{app}(q_\pm) + \tilde{Y}^+_{app}(q_\pm) - 2Y^\pm_{app}(q_\pm)}{\tilde{Y}^-_{app}(q_\pm) - \tilde{Y}^+_{app}(q_\pm)} \right). \tag{28}$$

Thus, the approximate radial wave function is determined completely.

IV. NUMERICAL ILLUSTRATIONS

Now we present some graphic and numerical illustrations which demonstrate properties of the quasi-harmonic approximation in the case of Cornell potential. First, Fig. 2 and Fig. 3 show the normalized continuous radial wave functions. Solid lines correspond to $k = 1$ and dashed lines correspond to $k = 0.2$.

Further we calculate the expectation values

$$e_{app} = \langle \psi_{app} | \hat{H} | \psi_{app} \rangle \tag{29}$$

of the Hamiltonian

$$\hat{H} = -\frac{d^2}{dx^2} - \frac{k}{x} + x + \frac{l(l+1)}{x^2}. \tag{30}$$

with the help of the normalized approximate wave functions. We compare our results e_{app} with results e_{var} of scaling variational method [2] and with results e_{num} of numerical integration of the Schrödinger equation [1], which are considered as exact. This comparison is presented in Table I.

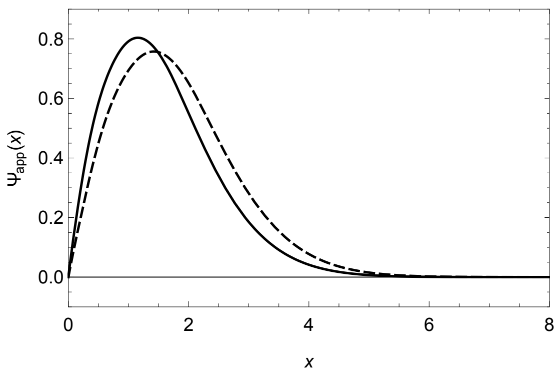

FIG. 2: Radial wave functions for $l = 0$, $n = 0$.

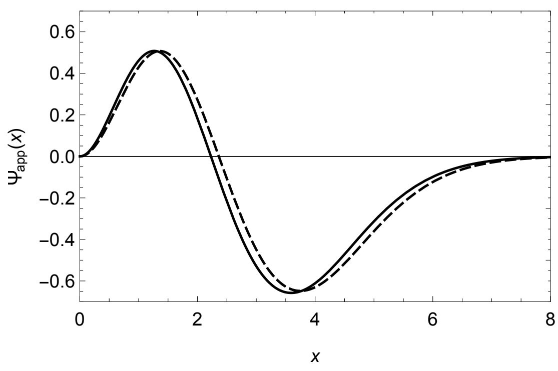

FIG. 3: Radial wave functions for $l = 1$, $n = 1$.

TABLE I: Energy levels.

n	l	e_{var}	e_{num}	e_{app}
		$k = 0.2$		
0	0	2.2913	2.1673	2.1782
1	0	3.8158	3.9702	3.9695
0	1	3.3726	3.2582	3.2643
1	1	4.8080	4.8019	4.8026
0	2	4.2759	4.1703	4.1757
1	2	5.6396	5.5634	5.5641
		$k = 1$		
0	0	1.4735	1.3979	1.4087
1	0	3.3370	3.4751	3.4731
0	1	2.9205	2.8255	2.8325
1	1	4.4585	4.4619	4.4626
0	2	3.9455	3.8506	3.8562
1	2	5.3585	5.2930	5.2937

V. CONCLUSION

So, we see the quasi-harmonic approximation with Langer's exponetial substitution yields the satisfactory description of the quarkonium bound states.

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