Two Coulomb centers problem in the Lobachevsky space for large intercenter separations

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The quantum mechanical problem of motion of a charged particle in the field of two Coulomb centers in the three-dimensional Lobachevsky space is studied in the case of large intercenter distance. Solutions of the separated Schrödinger equation are found in the form of series of hypergeometric functions. Asymptotic expressions for the energy levels are presented.

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I. INTRODUCTION

Models based on the use of solutions of the quantum mechanical Kepler – Coulomb problem in spaces of constant curvature have been applied for phenomenological description of quarconium spectra [1] and excitons in quantum dots [2]. This problem was first treated by Schrödinger [3] for the case of positive curvature and by Infeld and Schild [4] for negative curvature. Since the first works, great number of papers have appeared which are devoted to different aspects of this problem including symmetry and separation of variables (see e.g. [5]–[8]).

High symmetry related to the existence of conserved analog of the Runge – Lenz vector enables the separation of variables also in the problem of two Coulomb centers in spaces of constant curvature [9].

In this note we consider solutions of the Schrödinger equation with a potential of two Coulomb centers which are suitable for treatment of the problem in the case of large intercenter distances.

II. SEPARATION OF VARIABLES

We use the embedding of the Lobachevsky space ${}^1S_3$ in a four-dimensional pseudo-Euclidean space with coordinates x_μ, $\mu = 0, 1, 2, 3$. Then for points of the Lobachevsky space we have $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 = x_0^2 - \mathbf{x}^2 = \rho^2$, where $x = (x_0, x_1, x_2, x_3)$, $\mathbf{x} = (x_1, x_2, x_3)$, and constant ρ denotes the curvature radius.

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The Schrödinger equation for a particle moving in the potential field U in the Lobachevsky space we write in the form

$$H\Psi = E\Psi, \quad H = -\frac{1}{2}\Delta + U, \tag{1}$$

where Δ is the Laplacian operator in ${}^1S_3$. We use units such that $\hbar = m = 1$, where m is the mass of the particle.

The Coulomb potential of a charge Z located at the point ${}^0x = (\rho, 0, 0, 0)$ we take in the form $U = -Zx_0/|\mathbf{x}|$. Such a choice leads to a similarity between some expressions relating to spaces of positive and negative curvature. In particular, energy spectrum is given by the formula

$$E_n = -\frac{Z^2}{2n^2} - \frac{n^2-1}{2\rho^2}, \quad n = 1, 2, \ldots,$$

which can be obtained from the corresponding formula for the three-dimensional sphere by a substitution $R^2 \to -\rho^2$, where R denotes the radius of the sphere.

The expression for the Coulomb potential of a charge Z_1 located at the point ${}^1x = ({}^1x_0, {}^1x_1, {}^1x_2, {}^1x_3)$ is then as follows:

$$U_1 = -\frac{Z_1}{\rho}\frac{({}^1xx)}{\sqrt{({}^1xx)^2 - \rho^4}},$$

where $({}^1xx) = {}^1x_0x_0 - {}^1x_1x_1 - {}^1x_2x_2 - {}^1x_3x_3$.

Now we introduce coordinates s, t, φ related to the Cartesian coordinates x_μ as follows:

$$x_0 = \frac{\rho\gamma\sqrt{\gamma^2-1}}{\sqrt{(\gamma^2-t^2)(\gamma^2-s^2)}}, \; x_1 = \rho\gamma\sqrt{\frac{(t^2-1)(1-s^2)}{(\gamma^2-t^2)(\gamma^2-s^2)}}\cos\varphi,$$

$$x_2 = \rho\gamma\sqrt{\frac{(t^2-1)(1-s^2)}{(\gamma^2-t^2)(\gamma^2-s^2)}}\sin\varphi, \; x_3 = \frac{\rho st\sqrt{\gamma^2-1}}{\sqrt{(\gamma^2-t^2)(\gamma^2-s^2)}},$$
$$1 \le t < \gamma, \; -1 \le s \le 1, \; 0 \le \varphi < 2\pi,$$

where γ is some constant.

The metrics of the Lobachevsky space in these coordinates takes the form

$$dl^2 = \frac{\rho^2\gamma^2(\gamma^2-1)(t^2-s^2)dt^2}{(t^2-1)(\gamma^2-s^2)(\gamma^2-t^2)^2} + \frac{\rho^2\gamma^2(\gamma^2-1)(t^2-s^2)ds^2}{(1-s^2)(\gamma^2-s^2)^2(\gamma^2-t^2)} + \frac{\rho^2\gamma^2(t^2-1)(1-s^2)d\varphi^2}{(\gamma^2-s^2)(\gamma^2-t^2)},$$

and the Laplacian operator is

$$\Delta = \frac{(\gamma^2-s^2)(\gamma^2-t^2)}{\rho^2\gamma^2(\gamma^2-1)(t^2-s^2)}\left[(\gamma^2-t^2)\frac{\partial}{\partial t}(t^2-1)\frac{\partial}{\partial t}\right.$$
$$\left. + (\gamma^2-s^2)\frac{\partial}{\partial s}(1-s^2)\frac{\partial}{\partial s}\right] + \frac{(\gamma^2-s^2)(\gamma^2-t^2)}{\rho^2\gamma^2(t^2-1)(1-s^2)}\frac{\partial^2}{\partial\varphi^2}.$$

If the Coulomb charges Z_1 and Z_2 are located at the points of the Lobachevsky space with coordinates $t = 1$, $s = -1$ and $t = 1$, $s = 1$, respectively, then the distance between charges is

$$\mathcal{D} = \rho \ln\frac{\gamma+1}{\gamma-1},$$

and their potential is given by expression

$$U=-\frac{Z_1(\gamma^2+st)}{\rho\gamma(s+t)}-\frac{Z_2(\gamma^2-st)}{\rho\gamma(t-s)}.$$

Separating the Schrödinger equation with this potential with the help of the substitution $\Psi=u(t)v(s)e^{\pm m\varphi}$, $m=0,\,1,\,2,...$, one obtains ordinary differential equations

$$(\gamma^2-t^2)\frac{d}{dt}(t^2-1)\frac{du}{dt}+\left[\frac{2Z_+\gamma\rho(\gamma^2-1)t}{\gamma^2-t^2}+\frac{2E\gamma^2\rho^2(t^2-1)}{\gamma^2-t^2}-\frac{m^2(\gamma^2-t^2)}{t^2-1}-\Lambda\right]u=0, \quad (2)$$

$$(\gamma^2-s^2)\frac{d}{ds}(1-s^2)\frac{d}{ds}+\left[\frac{2Z_-\gamma\rho(\gamma^2-1)s}{\gamma^2-s^2}+\frac{2E\gamma^2\rho^2(1-s^2)}{\gamma^2-s^2}-\frac{m^2(\gamma^2-s^2)}{1-s^2}+\Lambda\right]v=0, \quad (3)$$

where $Z_\pm=Z_2\pm Z_1$.

In the limit of vanishing curvature we have $\rho\to\infty$ and $\mathcal{D}\to d$, where d is the distance between charges in flat space. Assuming that in this limit $\Lambda/\gamma^2\to\lambda$, we obtain the limiting form of equations (2) and (3):

$$\frac{d}{dt}(t^2-1)\frac{du}{dt}+\left[at-p^2(t^2-1)-\frac{m^2}{t^2-1}-\lambda\right]u=0, \quad (4)$$

$$\frac{d}{ds}(1-s^2)\frac{d}{ds}+\left[bs-p^2(1-s^2)-\frac{m^2}{1-s^2}+\lambda\right]v=0, \quad (5)$$

where $p^2=-Ed^2/2$, $a=Z_+d$, $b=Z_-d$. Equations (4) and (5) are named usually quasiradial and quasiangular equation, respectively. We will also use these names for their analogs in the Lobachevsky space, equations (2) and (3).

III. SOLUTIONS OF THE QUASIRADIAL EQUATION

Each of the equations (2) and (3) has four regular singularities. Substitutions

$$x=\frac{2\gamma(t-1)}{(\gamma-1)(t+\gamma)} \quad (6)$$

and

$$u=(x)^{m/2}(x-1)^{\sigma_+}(x-a)^{m/2}g, \quad (7)$$

where

$$\sigma_\pm=\frac{1}{2}\left(1+\sqrt{-2E\rho^2\mp 2Z_+\rho+1}\right),$$

result in the Heun equation for g:

$$\frac{d^2g}{dx^2}+\left(\frac{m+1}{x}+\frac{2\sigma_+}{x-1}+\frac{m+1}{x-a}\right)\frac{dg}{dx}+\frac{(Ax-p)g}{x(x-1)(x-a)}=0, \quad (8)$$

where

$$a=-\frac{4\gamma}{(\gamma-1)^2},\; p=(\sigma_+a+m)(m+1)-\frac{\rho Z_+a}{2}+\frac{\Lambda}{4\gamma},$$
$$A=(\sigma_++\sigma_-+m)(\sigma_+-\sigma_-+m+1).$$

The case of large distances between centers corresponds to values of $\gamma\sim 1$ and large values of a. When $a\to\infty$, the equation (8) turns into the hypergeometric equation. Therefore we will look for solutions of the equation (8) at large intercenter separations in the form of series of hypergeometric functions. Here we only consider bound states of the system. Thus, in order to obtain the solution of equation (2) bounded for $1 \leq t < \gamma$, we write the corresponding solution of equation (8) as a series

$$g = \sum_{k=-n_1}^{\infty} c_k \, {}_2F_1(-n_1 - k, n_1 + k + m + 2\sigma_+; m + 1; x) \tag{9}$$

with integer n_1. Substituting (9) in (8) we obtain recurrence relations for c_k

$$\alpha_k c_{k+1} + \beta_k c_k + \gamma_k c_{k-1} = 0, \tag{10}$$

where

$$\begin{aligned}
\alpha_k = \frac{(n_1 + k + 1)(n_1 + k - \sigma_- + \sigma_+ + 1)(n_1 + k + \sigma_- + \sigma_+)(n_1 + k + 2\sigma_+)}{(2n_2 + 2k + m + 2\sigma_+ + 1)(2n_2 + 2k + m + 2\sigma_+ + 2)}, \\
\beta_k = \left(a - \frac{1}{2}\right)\left[\frac{\Lambda}{4\gamma} + \frac{E\rho^2}{2} + (n_1 + \sigma_+ + k + m)(n_1 + \sigma_+ + k)\right] \\
+ \frac{m^2 - 1}{8} + \frac{\Lambda}{8\gamma} - \frac{E\rho^2}{4} \\
- \frac{[(2\sigma_+ - 1)^2 - m^2][(2\sigma_- - 1)^2 - m^2]}{8(2n_1 + 2k + m + 2\sigma_+ + 1)(2n_1 + 2k + m + 2\sigma_+ - 1)}, \\
\gamma_k = (n_1 + k + m)(n_1 + k + m - \sigma_- + \sigma_+) \\
\times \frac{(n_1 + k + m + \sigma_- + \sigma_+ - 1)(n_1 + k + m + 2\sigma_+ - 1)}{(2n_1 + 2k + m + 2\sigma_+ - 1)(2n_1 + 2k + m + 2\sigma_+ - 2)}.
\end{aligned} \tag{11}$$

Using relations (10), (11) we obtain an expansion of the separation constant Λ in powers of $\varepsilon = -1/a$:

$$\begin{aligned}
\Lambda/\gamma = -4(n_1 + \sigma_+)(n_1 + \sigma_+ + m) - 2E\rho^2 \\
-\varepsilon\left(\frac{1 - m^2}{2} + 2E\rho^2 + 2(n_1 + \sigma_+)(n_1 + \sigma_+ + m)\right. \\
\left. + \frac{[(2\sigma_+ - 1)^2 - m^2][(2\sigma_- - 1)^2 - m^2]}{(2n_1 + m + 2\sigma_+ + 1)(2n_1 + m + 2\sigma_+ - 1)}\right) \\
+4\varepsilon^2\left(\frac{(n_1 + 1)(n_1 + m + 1)(2\sigma_+ + n_1)(m + 2\sigma_+ + n_1)}{(m + 2\sigma_+ + 2n_1)(m + 2\sigma_+ + 2n_1 + 1)^3(m + 2\sigma_+ + 2n_1 + 2)}\right. \\
\times b_+(b_- + 1)(b_- + m + 1)(b_+ + m) \\
-b_-(m + n_1)(b_- + m)(b_+ - 1)(b_+ + m - 1) \\
\left.\times \frac{n_1(m + n_1)(2\sigma_+ + n_1 - 1)(m + 2\sigma_+ + n_1 - 1)}{(m + 2\sigma_+ + 2n_1 - 2)(m + 2\sigma_+ + 2n_1 - 1)^3(m + 2\sigma_+ + 2n_1)}\right) + O(\varepsilon^3),
\end{aligned} \tag{12}$$

where $b_\pm = n_1 + \sigma_+ \pm \sigma_-$.

IV. SOLUTIONS OF THE QUASIANGULAR EQUATION AND ENERGY SPECTRUM

Making substitutions

$$z = \frac{2\gamma(s + 1)}{(\gamma - 1)(s - \gamma)}, \ y = \frac{2\gamma(s - 1)}{(\gamma - 1)(s + \gamma)} \tag{13}$$

and transformations of the dependent variable,

$$v = z^{m/2}(1-z)^{\mu_+}(z-a)^{m/2} f \tag{14}$$

or

$$v = y^{m/2}(1-y)^{\mu_-}(y-a)^{m/2} f', \tag{15}$$

where

$$\mu_\pm = \frac{1}{2}\left(1 - \sqrt{-2E\rho^2 \mp 2Z_-\rho + 1}\right),$$

we obtain equations of the Heun type for f and f'

$$\frac{d^2 f}{dz^2} + \left(\frac{m+1}{z} + \frac{2\mu_+}{z-1} + \frac{m+1}{z-a}\right)\frac{df}{dz} + \frac{Bz-q}{z(z-1)(z-a)} f = 0, \tag{16}$$

$$\frac{d^2 f'}{dy^2} + \left(\frac{m+1}{y} + \frac{2\mu_-}{y-1} + \frac{m+1}{y-a}\right)\frac{df'}{dy} + \frac{B'y-q'}{y(y-1)(y-a)} f' = 0, \tag{17}$$

where

$$\begin{gathered} q = (m+1)(\mu_+ a + m) - \frac{\rho Z_- a}{2} + \frac{\Lambda a}{4\gamma},\ q' = (m+1)(\mu_- a + m) + \frac{\rho Z_- a}{2} + \frac{\Lambda a}{4\gamma}, \\ B = (\mu_+ + \mu_- + m)(\mu_+ - \mu_- + m + 1),\ B' = (\mu_+ + \mu_- + m)(\mu_- - \mu_+ + m + 1). \end{gathered} \tag{18}$$

We will consider solutions of equation (3) of the form

$$v_1 = \phi(z, \mu_+) \sum_{k=-\infty}^{\infty} d_k\, {}_2F_1(-\nu_2 - k, m + \nu_2 + k + 2\mu_+; m+1; z), \tag{19}$$

$$v_1' = \phi(y, \mu_-) \sum_{k=-\infty}^{\infty} d_k\, {}_2F_1(-\nu_2' - k, m + \nu_2' + k + 2\mu_-; m+1; y), \tag{20}$$

where

$$\phi(x, \mu) = (-x)^{m/2}(1-x)^{\mu}(x-a)^{m/2}, \qquad \nu_2' = \nu_2 + \mu_+ - \mu_-, \tag{21}$$

and ν_2 is a parameter. Coefficients d_k satisfy the recurrence relation

$$\alpha_k^\nu d_{k+1} + \beta_k^\nu d_k + \gamma_k^\nu d_{k-1} = 0, \tag{22}$$

where

$$\begin{aligned} \alpha_k^\nu &= \frac{(\nu_2 + k + 1)(\nu_2' + k + 1)(\nu_2' + 2\mu_- + k)(\nu_2 + 2\mu_+ + k)}{(2\nu_2 + 2\mu_+ + 2k + m + 1)(2\nu_2 + 2\mu_+ + 2k + m + 2)}, \\ \beta_k^\nu &= \left(a - \frac{1}{2}\right)\left[\frac{\Lambda}{4\gamma} + \frac{E\rho^2}{2} + (\nu_2 + \mu_+ + k + m)(\nu_2 + \mu_+ + k)\right] \\ &\quad + \frac{m^2 - 1}{8} + \frac{\Lambda}{8\gamma} - \frac{E\rho^2}{4} \\ &\quad - \frac{[(2\mu_+ - 1)^2 - m^2][(2\mu_- - 1)^2 - m^2]}{8(2\nu_2 + 2\mu_+ + 2k + m + 1)(2\nu_2 + 2\mu_+ + 2k + m - 1)}, \\ \gamma_k^\nu &= (\nu_2 + k + m)(\nu_2' + k + m) \\ &\quad \times \frac{(\nu_2 + 2\mu_+ + k + m - 1)(\nu_2' + 2\mu_- + k + m - 1)}{(2\nu_2 + 2\mu_+ + 2k + m - 1)(2\nu_2 + 2\mu_+ + 2k + m - 2)}. \end{aligned} \tag{23}$$

From relations (22), (23) we obtain an expansion of the separation constant Λ in powers of $\varepsilon = -1/a$:

$$\begin{aligned}
\Lambda/\gamma = -4(\nu_2+\mu_+)(\nu_2+\mu_++m) - 2E\rho^2 \\
-\varepsilon\left(\frac{1-m^2}{2} + 2E\rho^2 + 2(\nu_2+\mu_+)(\nu_2+\mu_++m)\right. \\
\left.+\frac{[(2\mu_+-1)^2-m^2][(2\mu_--1)^2-m^2]}{(2\nu_2+m+2\mu_++1)(2\nu_2+m+2\mu_+-1)}\right) \\
+4\varepsilon^2\left(\frac{(\nu_2'+2\mu_-)(\nu_2'+2\mu_-+m)(2\mu_++\nu_2)(m+2\mu_++\nu_2)}{(m+2\mu_++2\nu_2)(m+2\mu_++2\nu_2+1)^3(m+2\mu_++2\nu_2+2)}\right. \\
\times(\nu_2+1)(\nu_2+m+1)(\nu_2'+1)(\nu_2'+m+1) \\
-\nu_2\nu_2'(\nu_2+m)(\nu_2'+m) \\
\left.\times\frac{(\nu_2'+2\mu_--1)(\nu_2'+2\mu_-+m-1)(2\mu_++\nu_2-1)(m+2\mu_++\nu_2-1)}{(m+2\mu_++2\nu_2-2)(m+2\mu_++2\nu_2-1)^3(m+2\mu_++2\nu_2)}\right) + O(\varepsilon^3) \quad (24)
\end{aligned}$$

By comparing expressions for Λ (12) and (24) we can derive an expansion for values of energy which still includes parameter ν_2. This parameter is determined by the requirement of finiteness of angular functions $v(s)$ for $-1 \le s \le 1$. Function v_1 (19) is finite at $s=-1$, and function v_1' (20) is finite at $s=1$. Solutions v_1 and v_1' are supposed to represent the same wave function. In order to match these solutions, we present them as linear combinations of further solutions of equation (3)

$$v_1 = \Gamma(m+1)(v_2+v_3), \qquad v_1' = \Gamma(m+1)(v_2'+v_3'), \quad (25)$$

where

$$\begin{aligned}
v_2 &= \phi(z,\mu_+)\sum_{k=-\infty}^{\infty}\frac{d_k\Gamma(2\nu_2+2k+2\mu_++m)(-z)^{\nu_2+k}}{\Gamma(\nu_2+k+2\mu_++m)\Gamma(\nu_2+k+m+1)} \\
&\times\ {}_2F_1(-\nu_2-k,-\nu_2-k-m;1-2\nu_2-2k-2\mu_+-m;1/z), \\
v_3 &= (1-z)^{1-2\mu_+}\phi(z,\mu_+) \\
&\times\sum_{k=-\infty}^{\infty}\frac{d_k\Gamma(-2\nu_2-2k-2\mu_+-m)(-z)^{-\nu_2-k-m-1}}{\Gamma(-\nu_2-k-2\mu_++1)\Gamma(-\nu_2-k)} \\
&\times\ {}_2F_1(\nu_2+k+1,\nu_2+k+m+1;1+2\nu_2+2k+2\mu_++m;1/z), \quad (26) \\
v_2' &= \phi(y,\mu_-)\sum_{k=-\infty}^{\infty}\frac{d_k\Gamma(2\nu_2'+2k+2\mu_-+m)(-y)^{\nu_2'+k}}{\Gamma(\nu_2'+k+2\mu_-+m)\Gamma(\nu_2'+k+m+1)} \\
&\times\ {}_2F_1(-\nu_2'-k,-\nu_2'-k-m;1-2\nu_2'-2k-2\mu_--m;1/y), \\
v_3' &= (1-y)^{1-2\mu_-}\phi(y,\mu_-) \\
&\times\sum_{k=-\infty}^{\infty}\frac{d_k\Gamma(-2\nu_2'-2k-2\mu_--m)(-y)^{-\nu_2'-k-m-1}}{\Gamma(-\nu_2'-k-2\mu_-+1)\Gamma(-\nu_2'-k)} \\
&\times\ {}_2F_1(\nu_2'+k+1,\nu_2'+k+m+1;1+2\nu_2'+2k+2\mu_-+m;1/y). \quad (27)
\end{aligned}$$

If a closed circuit is described in the complex plane of s which makes positive loops around points $s=1$ and $s=\gamma$, then from equation (13) we have

$$y \to e^{4\pi i}y, \qquad 1-y \to e^{4\pi i}(1-y), \qquad z \to e^{-4\pi i}z, \qquad 1-z \to e^{-4\pi i}(1-z),$$

and it is easily seen that the effect of this circulation on the solutions (26), (27) of equation (3) is

$$v_2 \to \exp[-4\pi i(\nu_2+\mu_+)]v_2 \qquad v_2' \to \exp[4\pi i(\nu_2+\mu_+)]v_2'$$
$$v_3' \to \exp[-4\pi i(\nu_2+\mu_+)]v_3' \qquad v_3 \to \exp[4\pi i(\nu_2+\mu_+)]v_3. \tag{28}$$

Since v_2, v_2', v_3, v_3' are solutions of an ordinary differential equation of the second order, equation (28) implies that

$$v_3' = Kv_2, \qquad v_3 = K'v_2' \tag{29}$$

where K and K' are some constants.

From (25) and (29) it is seen that equation

$$KK' = 1 \tag{30}$$

is the necessary condition for solutions of equation (3) to be finite for $-1 \le s \le 1$. Expressions for K and K' can be obtained by expanding solutions (26), (27) into Laurent series of $1-y$ and $1-z$ and comparing like terms in relations (29). Thus we have

$$K = (1-a)^{-\mu_+-\nu_2-m/2}$$
$$\times \frac{\Gamma(\nu_2+m+1)\Gamma(\nu_2+2\mu_++m)\Gamma(1-2\nu-2\mu_+-m)}{\Gamma(-\nu_2')\Gamma(1-\nu_2'-2\mu_-)\Gamma(1+2\nu_2+2\mu_++m)}(1+O(\varepsilon))$$
$$K' = (1-a)^{-\mu_+-\nu_2-m/2}$$
$$\times \frac{\Gamma(\nu_2'+m+1)\Gamma(\nu_2'+2\mu_-+m)\Gamma(1-2\nu-2\mu_+-m)}{\Gamma(-\nu_2)\Gamma(1-\nu_2-2\mu_+)\Gamma(1+2\nu_2+2\mu_++m)}(1+O(\varepsilon)) \tag{31}$$

Now we can find from equation (30) that finiteness of angular functions requires the choice $\nu_2 = n_2+\delta$ or $\nu_2' = n_2'+\delta$, where n_2 (or n_2') is an integer, and $\delta = o(\varepsilon)$. We take n_2 integer, which corresponds to localization of charged particle near the center Z_1. With this assumption, we can write an expansion for energy levels of bound states in powers of $\varepsilon = -1/a$

$$E = E^{(0)} + \varepsilon E^{(1)} + \varepsilon^2 E^{(2)} + ..., \quad E^{(0)} = -\frac{Z_1^2}{2n^2} - \frac{n^2-1}{2\rho^2} - \frac{Z_2}{\rho},$$
$$E^{(1)} = \frac{2Z_1Z_2(n^4-Z_1^2\rho^2)}{(-n+n\Delta+Z_1\rho)(n\Delta+Z_1\rho)(n+n\Delta+Z_1\rho)}$$
$$\approx -\frac{2Z_2}{\rho}\left[1-\frac{3n\Delta}{\rho Z_1}-\frac{n^2(n^2-6\Delta^2-1)}{\rho^2 Z_1^2}\right],$$
$$E^{(2)} \approx \frac{nZ_2}{\rho^2 Z_1}\left[6\Delta+\frac{6n(n^2-6\Delta^2-1)}{\rho Z_1}\right.$$
$$\left.-\frac{n^2(-129Z_1\Delta^3-3nZ_2\Delta^2+3\Delta(15n^2+3m^2-23)Z_1+n(17n^2-9m^2+19)Z_2)}{\rho^2 Z_1^2}\right], \tag{32}$$

where $n = n_1+n_2+|m|+1$ and $\Delta = n_1-n_2$. The general expression for $E^{(2)}$ for a state with arbitrary quantum numbers is too cumbersome, and we only give here first terms of the expansion in powers of ρ^{-1}.

With the help of equation (32) we can define δ more exactly, namely

$$\delta = O(\varepsilon^{n+Z_1\rho/n-2n_2-m-1}).$$

Since bound states localized near Z_1 exist only for $n^2 < Z_1\rho$, asymptotic expansion for the energy values in the form (32) is valid for all levels but the highest.

V. CONCLUSION

We found solutions of the separated Schrödinger equation for a charged particle in the field of two Coulomb centers in the Lobachevsky space in the form of series of hypergeometric functions. The series solution of the quasiradial equation converges in all the range of physical variable. Finite solutions of the quasiangular equation were obtained by the use of relations between series convergent in different domains. This enabled us to derive asymptotic expressions for discrete energy levels of the charged particle in the case of large distance between the Coulomb centers in the Lobachevsky space.

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