Local Thermalization of Gluons in a Nonlinear Model

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Analytic solutions of a nonlinear boson diffusion equation account for the fast local equilibration of gluons in relativistic heavy-ion collisions using schematic initial conditions. The solutions describe the time-dependent approach to the Bose-Einstein equilibrium distribution with a local equilibration time of $\tau_{\rm eq} \simeq 0.1$ fm/c and central temperatures of the order of $500-600$ MeV in the initial stages of Pb-Pb collisions at energies reached at the Large Hadron Collider (LHC).

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I. INTRODUCTION

The fast local thermalization of gluons in the initial stages of relativistic heavy-ion collisions is a prerequisite for hydrodynamic descriptions [1] of the subsequent collective expansion and cooling of the hot fireball that is created in the collision. Typical local equilibration times for gluons are about 0.1 fm/c [2], with initial central temperatures in a Pb-Pb collision at $\sqrt{s_{\rm NN}} = 5.02$ TeV reached at the Large Hadron Collider (LHC) of the order of 500-600 MeV [3].

Whereas numerical calculations relying on a Boltzmann collision term are available to account for the initial local equilibration, it is of interest to have an exactly solvable analytic model to better understand the physics of the fast equilibration. A corresponding nonlinear boson diffusion equation (NBDE) has been presented in Ref. [4]. It preserves the essential features of Bose-Einstein statistics that are contained in the collision term. As a consequence of its nonlinearity, the thermal equilibrium distribution emerges as a stationary solution and hence, the equation appears suitable to model the thermalization of gluons in relativistic collisions.

For a given initial nonequilibrium gluon distribution at $t = 0$, solutions of the nonlinear boson diffusion equation will then describe the time-dependent equilibration towards the thermal distribution with the local temperature T. In Ref. [4] such solutions were, however, calculated with the free Green's function. Whereas this accounts for local thermalization in the ultraviolet (UV) with the corresponding equilibration time $\tau_{\rm eq}$, in the infrared (IR) the populations decrease due to diffusion into the negative-energy region.

Such an unphysical behaviour is avoided if one considers the boundary condition at the singularity $|\mathbf{p}| = \mathrm{p} = \epsilon = \mu$ with the chemical potential $\mu < 0$, and the corresponding bounded Green's function in the solution of the NBDE. With this Green's function, populations indeed attain the B-E limit also in the infrared for nonequilibrium initial conditions that include the singularity.

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The nonlinear model and the solution of the combined initial- and boundary value problem are briefly reviewed in the next section. Subsequently, the thermalization problem is solved for a schematic initial gluon distribution that characterizes the relativistic collision at $t = 0$. Adding the boundary condition at the singularity, $n(\epsilon = \mu < 0, t) \to \infty$, the time-dependent partition function that includes initial and boundary conditions is obtained using analytic expressions for both, the bound Green's function, and the function that contains an integral over the initial conditions. The resulting occupation-number distribution function $n(\epsilon, t)$ is calculated, and it is shown to approach the equilibrium distribution both in the UV, and in the IR.

II. NONLINEAR BOSON DIFFUSION EQUATION AND SOLUTION

The transport equation (NBDE) for the single-particle occupation probability distributions $n \equiv n_{\rm th}(\epsilon, t)$ has been derived from the bosonic Boltzmann collision term in Ref. [4]. The many-body physics is contained in the transport coefficients, which depend on energy, time, and the second moment of the interaction. The drift term $v(\epsilon, t)$ accounts for dissipative effects, the term $D(\epsilon, t)$ for diffusion of particles in the energy space. In the limit of energy-independent transport coefficients, however, the nonlinear boson diffusion equation for the occupation-number distribution $n(\epsilon, t)$ becomes

$$\frac{\partial n}{\partial t} = -v\,\frac{\partial}{\partial \epsilon}\Big[n\,(1+n)\Big] + D\,\frac{\partial^2 n}{\partial \epsilon^2}\,. \tag{1}$$

A stationary solution is given by the thermal distribution

$$n_{\rm eq}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/T} - 1} \tag{2}$$

with the chemical potential $\mu < 0$ in a finite boson system. In spite of its simple structure, the NBDE with constant transport coefficients preserves the essential features of Bose-Einstein statistics which are contained in the bosonic Boltzmann equation.

The transport equation can be solved exactly for a given initial condition $n_i(\epsilon)$ using the nonlinear transformation outlined in Ref. [4]. Before proceeding to the case of local gluon thermalization, the solution is briefly reconsidered. It can be written as

$$n(\epsilon,t) = -\frac{D}{v}\frac{\partial}{\partial\epsilon}\ln\mathcal{Z}(\epsilon,t) - \frac{1}{2} = -\frac{D}{v}\frac{1}{\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial\epsilon} - \frac{1}{2} \quad (3)$$

with the time-dependent partition function $\mathcal{Z}(\epsilon,t)$ obeying a linear diffusion equation

$$\frac{\partial}{\partial t}\mathcal{Z}(\epsilon,t) = D\frac{\partial^2}{\partial\epsilon^2}\mathcal{Z}(\epsilon,t)\,. \quad (4)$$

In the absence of boundary conditions, the free partition function becomes

$$\mathcal{Z}_{\text{free}}(\epsilon,t) = a(t)\int_{-\infty}^{+\infty} G_{\text{free}}(\epsilon,x,t)\,F(x)\,\mathrm{d}x\,, \quad (5)$$

but the occurence of the singularity requires the consideration of boundary conditions at $\epsilon = \mu < 0$. The partition function is unique up to multiplication with energy-independent prefactors $a(t)$: These drop out when taking the logarithmic derivative in the calculation of the occupation-number distribution. The initial conditions that are contained in the function $F(x)$ cover the full energy region $-\infty < x < \infty$.

For a solution without boundary conditions as in Refs. [4, 5], Green's function $G_{\text{free}}(\epsilon,x,t)$ of Eq. (4) is a single Gaussian

$$G_{\text{free}}(\epsilon,x,t) = \exp\Big(-\frac{(\epsilon-x)^2}{4Dt}\Big)\,, \quad (6)$$

but it becomes more involved once boundary conditions are considered. The function $F(x)$ depends on the initial occupation-number distribution n_{i} according to

$$F(x) = \exp\Big[-\frac{1}{2D}\big(vx + 2v\int_0^x n_{\text{i}}(y)\,\mathrm{d}y\big)\Big]\,. \quad (7)$$

The definite integral over the initial conditions taken at the lower limit $y = 0$ drops out in the calculation of $n(\epsilon,t)$ when performing the logarithmic derivative. Hence, the integral can be replaced [6] by the indefinite integral $A_{\text{i}}(x)$ over the initial distribution with $\partial_x A_{\text{i}}(x) = n_{\text{i}}(y)$, such that

$$F(x) = \exp\Big[-\frac{1}{2D}\left(vx + 2vA_{\text{i}}(x)\right)\Big]\,. \quad (8)$$

It is now possible to compute the partition function and the overall solution for the occupation-number distribution function Eq. (3) analytically, even in the presence of a singularity in the initial conditions – which had been excluded in the initial conditions, and hence, in the solutions given in Refs. [4, 5].

This replacement still provides the exact solution. For any given initial distribution n_{i}, it is now possible to compute the partition function and the overall solution for the occupation-number distribution function Eq. (3) analytically.

To obtain physically meaningful solutions not only in the UV, but also in the IR, one has to consider the boundary conditions at the singularity. The analytic solution technique has been developed in Refs. [6, 7] for the case of a cold bosonic atom gas that undergoes evaporative cooling. Here the approach is carried over to equilibrating gluons at relativistic energies.

To solve the problem with boundary conditions at the singularity, the chemical potential is treated as a fixed parameter. With $\lim_{\epsilon\downarrow\mu} n(\epsilon,t) = \infty\ \ \forall\ t$, one obtains $\mathcal{Z}(\mu,t) = 0$, and the energy range is restricted to $\epsilon \geq \mu$. This requires a new Green's function that equals zero at $\epsilon = \mu\ \forall\, t$. It can be written as

$$G(\epsilon,x,t) = G_{\text{free}}(\epsilon-\mu,x,t) - G_{\text{free}}(\epsilon-\mu,-x,t)\,, \quad (9)$$

and the partition function with this boundary condition becomes

$$\mathcal{Z}(\epsilon,t) = \int_0^\infty G(\epsilon,x,t)\,F(x+\mu)\,\mathrm{d}x\,. \quad (10)$$

The function F remains unaltered with respect to Eq. (8), except for a shift of its argument by the chemical potential. With a given initial nonequilibrium distribution n_{i}, the NBDE can now be solved including boundary conditions at the singularity. The solution is given by Eq. (3).

III. THERMALIZATION OF GLUONS IN RELATIVISTIC COLLISIONS

For massless gluons at the onset of a relativistic hadronic collision, an initial-momentum distribution $n_{\text{i}}(|\boldsymbol{p}|) \equiv n_{\text{i}}(p) = n_{\text{i}}(\epsilon)$ has been proposed by Mueller [8] based on Ref. [9]. It accounts, in particular, for the situation at the start of a relativistic heavy-ion collision [10]. It amounts to assuming that all gluons up to a limiting momentum Q_{s} are freed on a short time scale $\tau_0 \sim Q_{\text{s}}^{-1}$, whereas all gluons beyond Q_{s} are not freed. Thus the initial gluon-mode occupation in a volume V is taken to be a constant up to Q_{s},

$$n_{\text{i}}(\epsilon) = \theta(1-\epsilon/Q_{\text{s}})\,\theta(\epsilon)\,. \quad (11)$$

Typical gluon saturation momenta for a longitudinal momentum fraction carried by the gluon $x \simeq 0.01$ turn out to be of the order $Q_{\text{s}} \simeq 1\,\text{GeV}$ [11], which is chosen for the present model investigation.

Results for the gluon thermalization from $n_{\text{i}}(\epsilon)$ to $n_{\text{eq}}(\epsilon)$ according to Eq. (1) have been calculated in Ref. [4] for the free case, without considering boundary conditions at the singularity. As a consequence, diffusion into the negative-energy region occured, depleting the occupation in the infrared such that the asymptotic distribution differed from Bose-Einstein.

As a remedy, one has to extend the energy scale in Eq. (11) to $\mu \leq \epsilon < \infty$, and include the boundary conditions at the singularity $\epsilon = \mu < 0$. This will cause the time-dependent solutions of the NBDE to properly approach the thermal B-E distribution over the full energy scale as $t \to \infty$. The initial condition is thus modified to include the singularity at $\epsilon = \mu$ according to

$$n_{\rm i}(\epsilon) = \theta(1-\epsilon/Q_{\rm s})\,\theta(\epsilon) + \frac{1}{\exp\left(\frac{\epsilon-\mu}{T}\right)-1}\,\theta(-\epsilon)\,, \quad (12)$$

with the equilibrium temperature T that is attained upon completion of the local equilibration, and a chemical potential μ which is adjusted such that the two partial distributions match at $\epsilon = 0$, with $n_{\rm i}(0) = 1$.

The time-dependent partition function with the above initial condition can now be calculated using the bound Green's function Eq. (9), and the function $F(x)$ from Eq. (8). The latter contains an indefinite integral over the initial condition Eq. (12) that can be carried out to obtain (with $x \to x+\mu$ in the argument of $F(x)$ as required by the boundary conditions)

$$F(x) = \exp\left[\frac{-v(x+\mu)}{2D}\right] F_1(x) F_2(x) \quad (13)$$

with the auxiliary functions

$$F_1(x) = \Big(\exp(-\mu/T) - \exp\Big[\frac{-x-\mu}{T}\Big]\Big)\,\theta(-\mu-x) + \big(\exp(-\mu/T)-1\big)\big)\,\theta(x+\mu), \quad (14)$$

$$F_2(x) = \exp\Big[(-v/D)\theta(x+\mu)((Q_{\rm s}-x-\mu)\theta(x+\mu-Q_{\rm s}) + x+\mu)\Big]. \quad (15)$$

The function $F(x)$ and its argument $\ln[F(x)]$ are plotted in Fig. 1. Due to the singularity in its argument, $F(x)$ vanishes at the origin. It is continuous, but not differentiable at $x = Q_{\rm s} - \mu$, which holds the key to the equilibration at the UV boundary.

The Green's function of Eq. (9) that includes the IR boundary condition can explicitly be written as

$$G(\epsilon, x, t) = \exp\left[\frac{-(\epsilon-\mu-x)^2}{4Dt}\right] - \exp\left[\frac{-(\epsilon-\mu+x)^2}{4Dt}\right]. \quad (16)$$

With $F(x)$ and $G(\epsilon, x, t)$, the partition function $\mathcal{Z}(\epsilon, t)$ of Eq. (10) and its derivative $\partial\mathcal{Z}/\partial\epsilon$ can now be calculated, as well as the occupation-number distribution $n(\epsilon, t)$ from Eq. (3). The full calculation may in principle be carried out analytically, resulting in lengthy expressions containing many error functions and exponentials. In the case of initial conditions that are appropriate for evaporative cooling of atomic Bose gases at very low energy, we recently performed such an exact calculation including the boundary conditions at the singularity in Ref. [6]. For simplicity, I now prefer to compute the partition function and its derivative using the *NIntegrate* and *Derivative* routines of Mathematica.

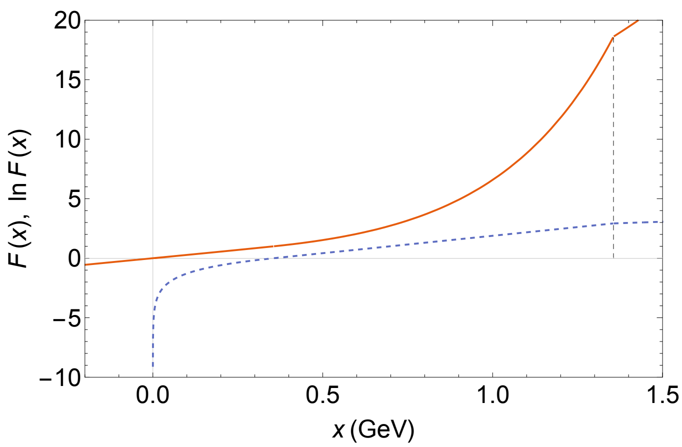

Figure 1. (color online) The function $F(x)$ of Eq. (8) (solid curve), and its argument $\ln F(x)$ (dashed curve) with $x \to x+\mu$ as required by the boundary conditions, and a singularity at the origin. $F(x)$ contains the integral over an initial nonequilibrium gluon distribution $n_{\rm i}(x)$ according to Eq. (13). The parameters are given in the text.

IV. DISCUSSION OF THE SOLUTIONS

Before presenting the nonlinear solutions, it is instructive to consider time-dependent solutions of the equilibration problem from $n_{\rm i}(\epsilon)$ to the thermal distribution $n_{\rm eq}(\epsilon)$ in the linear relaxation-time approximation (RTA), $\partial n_{\rm rel}/\partial t = (n_{\rm eq} - n_{\rm rel})/\tau_{\rm eq}$, given by

$$n_{\rm rel}(\epsilon, t) = n_{\rm i}(\epsilon)\,e^{-t/\tau_{\rm eq}} + n_{\rm eq}(\epsilon)(1-e^{-t/\tau_{\rm eq}})\,. \quad (17)$$

The bosonic equilibration time $\tau_{\rm eq}$ is taken as $\tau_{\rm eq} = 4D/(9v^2) \simeq 0.1$ fm/c. This expression has been determined in Ref. [4] for a θ-function initial distribution in the UV, provided the singularity at $\epsilon = \mu < 0$ is disregarded.

Time-dependent RTA-results are shown in Fig. 2 for $t = 0.02, 0.08, 0.15, 0.3$ and 0.6 fm/c. The thermal distribution with temperature $T = 513$ MeV is approached linearly, the discontinuities at $\epsilon = Q_{\rm s}$ persist. Thermalization for fixed $\tau_{\rm eq}$ occurs relatively slowly.

A more realistic account of the approach to equilibrium is provided by the solutions of the NBDE for the

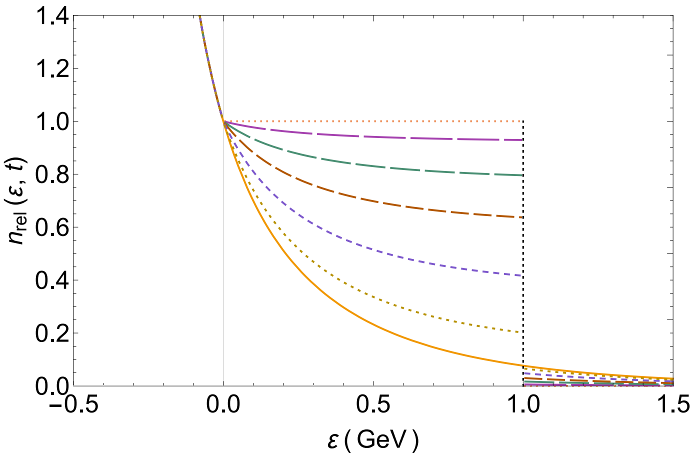

Figure 2. (color online) Local thermalization of gluons in the linear relaxation-time approximation (RTA) for $\mu < 0$. Starting from schematic initial conditions Eq. (12) in the cold system at $t = 0$ (box distribution with cut at $\epsilon = Q_s = 1$ GeV), a Bose-Einstein equilibrium distribution with temperature $T \simeq 513$ MeV (solid curve) is approached. Time-dependent single-particle occupation-number distribution functions are shown at $t = 0.02, 0.08, 0.15, 0.3$ and 0.6 fm/c (decreasing dash lenghts). Thermalization occurs much slower than in the nonlinear case, see Fig. 3.

gluon distribution functions, which are shown in Fig. 3 at $t = 6 \times 10^{-5}$, 6×10^{-4}, 6×10^{-3}, $4 \times 10^{-2}, 0.12$ and 0.36 fm/c, with decreasing dash lenghts. Thermalization occurs much faster than in the linear RTA case.

The steep cutoff in the UV at $\epsilon = Q_s$ is smeared out at short times – this was the case already in the free solution without boundary conditions [4]. The diffusion coefficient is $D = 1.17\,\mathrm{GeV}^2 c/\mathrm{fm}$, the drift coefficient $v = -2.28\,\mathrm{GeV}c/\mathrm{fm}$. Correspondingly, the equilibrium temperature in this model calculation is $T = -D/v \simeq 513$ MeV $= 0.513$ GeV, as expected for the initial central temperature in a Pb-Pb collision at the LHC energy of $\sqrt{s_{NN}} = 5$ TeV [3].

The bosonic local equilibration time in this calculation is the same as the one taken for the linear RTA result in Fig. 2, $\tau_{\mathrm{eq}} = 4D/(9v^2) \simeq 0.1$ fm/c. This is the time constant for reaching thermal equilibrium in the UV tail of the distribution function. It may take somewhat longer to attain equilibrium in the IR region, as is the case in the present model calculation. For given local temperature T and equilibration time τ_{eq}, the transport coefficients in the NBDE are obtained as

$$D = \frac{4}{9\,\tau_{\mathrm{eq}}}\,T^2\,, \qquad v = -\frac{4}{9\,\tau_{\mathrm{eq}}}\,T\,, \tag{18}$$

which for $\tau_{\mathrm{eq}} = 0.1$ fm/c corresponds to the values chosen above. The value of the gluonic chemical potential $\mu = -0.36$ GeV has been adapted such that for $T = 513$ MeV the initial condition at $\epsilon = 0$ becomes $n_{\mathrm{i}}(0) = 1$.

From Fig. 3 it is obvious that the inital nonequilibrium distribution $n_{\mathrm{i}}(\epsilon)$ gradually approaches the local thermal equilibrium $n_{\mathrm{eq}}(\epsilon)$ at $T = 513$ MeV through the solutions of the NBDE. As discussed already in Ref. [4], these solutions are expected to provide a more realistic description of the thermalization than the relaxation time approximation (RTA), which enforces a linear approach from n_{i} to n_{eq}, and cannot smoothen the initial discontinuities at the UV cutoff.

The assumption of a constant negative chemical potential $\mu < 0$ used in this work is, of course, an idealization that facilitates analytical solutions of the nonlinear problem. In general, the chemical potential is defined through the conservation of particle number, as is strictly fulfilled e.g. for atomic Bose gases. Driven by particle-number conservation, cold bosonic atoms can move into the condensed phase, thus diminishing the number of particles in the thermal cloud. The chemical potential in the equilibrium solution of the NBDE then becomes time dependent, as has been discussed in Ref. [6], albeit without a full quantum treatment of the condensed phase. It would become zero only in the limit of an inifinite number of particles in the condensed phase. Instead, it approaches a small but finite negative value for a finite number of particles.

In case of gluons in a relativistic heavy-ion collision, however, particle-number conservation is definitely not fulfilled, gluons can be created and destroyed. It is therefore unlikely that a condensed phase is actually formed, as had been proposed in model investigations where only soft elastic, number-conserving gluon collisions were considered [10]. Gluon condensate formation in relativistic collisions is essentially prevented by number-changing inelastic processes that correspond to splitting and merging of gluons, although a transient condensate formation is still being debated [12].

Hence, since inelastic collisions cannot be neglected, the gluon equilibrium distribution is expected to have a nearly vanishing, but still slightly negative chemical potential, which should be approached by the time-dependent solutions of the NBDE. It would therefore be of interest to repeat the present calculation for a time-dependent chemical potential, with $\mu(t) \to 0$ for $t \to \infty$, as was done in Ref. [6] for the case of cold atoms. This requires, however, numerical work that goes substantially beyond the analytic approach in the present note. As a first step, one can consider to repeat the present analytic calculation for the limiting case $\mu = 0$, replacing the initial condition of Eq. (12) with a box distribution plus a delta-function at $\epsilon = \mu = 0$, and boundary conditions at the singularity. For this special limiting case with $n_{\mathrm{eq}}(\epsilon = \mu \to 0) = \infty$, it should still be possible to obtain closed-form solutions of the NBDE.

V. CONCLUSION

Analytic solutions of the nonlinear boson diffusion equation have been explored for the thermalization of gluons in relativistic hadronic collisions. The solutions

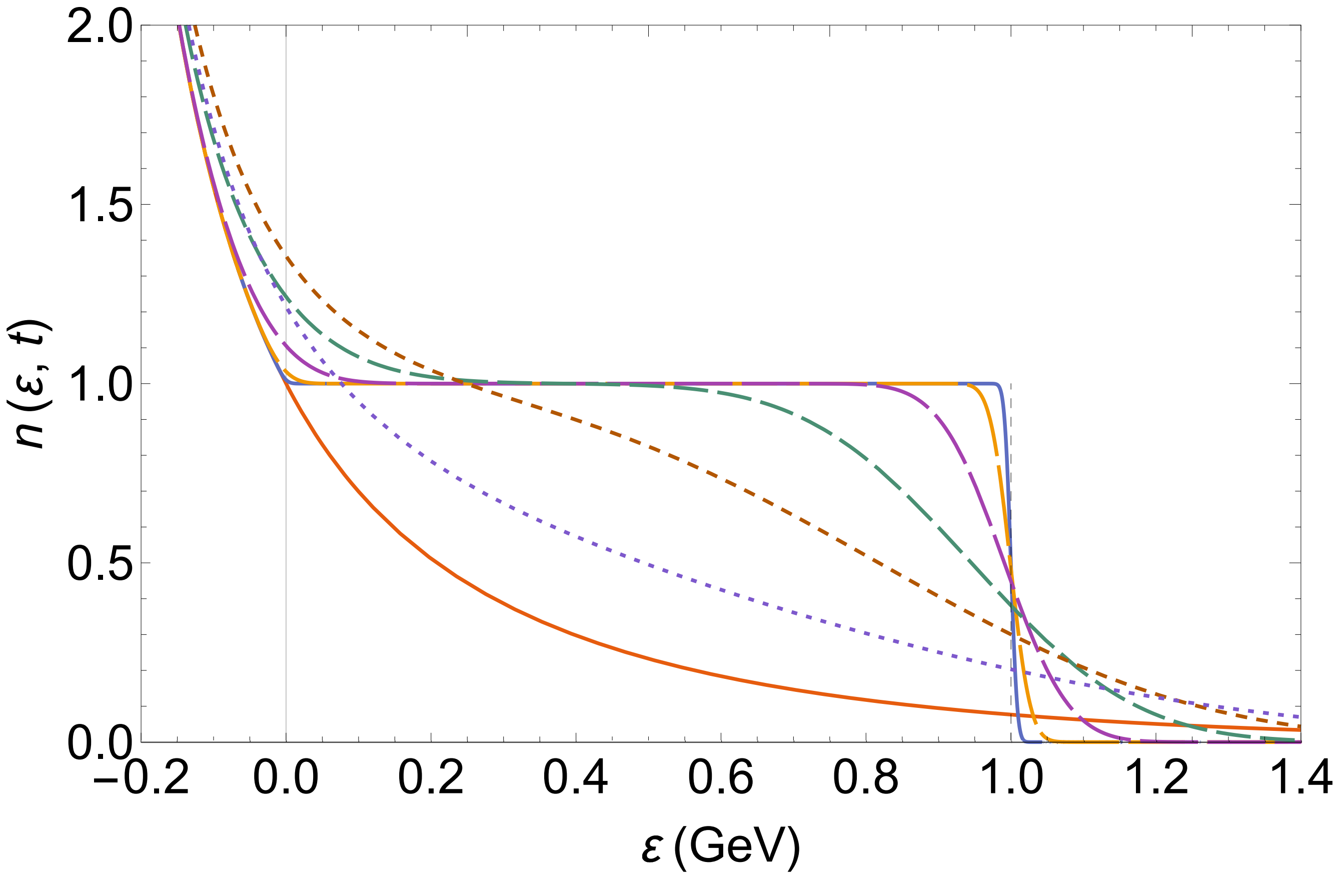

Figure 3. (color online) Local thermalization of gluons as represented by time-dependent solutions of the nonlinear boson diffusion equation (NBDE) for $\mu < 0$. Starting from schematic initial conditions Eq. (12) in the cold system at $t = 0$ (box distribution with cut at $\epsilon = Q_\mathrm{s} = 1$ GeV), a Bose-Einstein equilibrium distribution with temperature $T = 513$ MeV (lower solid curve) is approached. Time-dependent single-particle occupation-number distribution functions are shown at $t = 6 \times 10^{-5}$, 6×10^{-4}, 6×10^{-3}, 4×10^{-2}, 0.12 and 0.36 fm/c (decreasing dash lenghts).

take account of a singularity in the initial conditions at $\epsilon = \mu < 0$ with fixed chemical potential μ, and the boundary conditions at the singularity.

Different from earlier results that were calculated with the free Green's function and converged to the Bose-Einstein equilibrium solution only in the UV, these solutions converge towards B-E also in the IR and hence, properly describe thermalization in a finite gluon system.

The bounded solutions of the NBDE are, in particular, tailored to local thermalization processes that occur in relativistic heavy-ion collisions at energies reached at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider. In the present example, they are applied to the local equilibration of gluons in central Pb-Pb collisions at a center-of-mass energy of 5 TeV per nucleon pair, leading to rapid thermalization with a local temperature of $T \simeq 513$ MeV.

Since the thermalization occurs very fast – before anisotropic expansion fully sets in –, the analytic solution of the problem in 1+1 dimensions appears permissible. The hot system will subsequently expand anisotropically and cool rapidly, as is often modeled successfully by relativistic hydrodynamics [1], until hadronization is reached at $T \simeq 160$ MeV.

In conclusion, my schematic model based on the NBDE accounts for the fast nonlinear approach to local thermal equilibrium from an initial nonequilibrium gluon distribution at the start of the collision. It avoids the discontinuities that are inherent in the well-established relaxation time approximation, which enforces a linear approach to equilibrium.

Further refinements of the model such as time-dependent transport coefficients are conceivable, but are unlikely to allow for analytic solutions. A microscopic calculation of the transport coefficients with an investigation of their dependencies on energy and time would be very valuable. Extensions of the NBDE itself to higher dimensions in order to account for possible anisotropies should also be investigated.

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