On calculation of means of SDE

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Source SDE and main goal

Let use consider the stochastic differential equation, which in integral form can be rewritten as follows:

$$X_t = X_0 + \int_0^t \alpha(X_{s-})ds + \int_0^t \beta(X_{s-})dW_s, \quad t \in [0, T] \qquad (1)$$

where $X_0 \in \mathbb{R}$, W_t is the Wiener process. Here we suppose, that the integral on W_T in the right side of the equation (1) is the integral in Ito sense, and functions α and β are satisfies to Lipshitz condition, i.e. the equation has a strong solution (see, e.g. [1, 2]).

$$\mathbb{E}[X_t] = ?$$

Another representation of source SDE

Let us consider the process $X_{(\cdot)}$ at the time $t+\Delta t$. By the additivity of integrals we have:

$$X_{t+\Delta t} = X_0 + \int_0^t \alpha(X_{s-})ds + \int_0^t \beta(X_{s-})dW_s \tag{2}$$

$$+ \int_t^{t+\Delta t} \alpha(X_{s-})ds + \int_t^{t+\Delta t} \beta(X_{s-})dW_s, \tag{3}$$

Now, let us denote $\hat{X}_0 = X_t$, than

$$\hat{X}_{\Delta t} = \hat{X}_0 + \int_0^{\Delta t} \alpha(\hat{X}_s)ds + \int_0^{\Delta t} \beta(\hat{X}_s)dW_s. \tag{4}$$

Moments of SDE

Probability distribution function of $\hat{X}_t$ can be restored by its moments.
n-th moment can be calculated as the product of the right side of the equation (4).

By the properties of the $\hat{X}_{(\cdot)}$ the $\hat{X}_0$ and W_t are independent, therefore:

$$\mathbb{E}[f(\hat{X}_0, \hat{X}_{(\cdot)})] = \mathbb{E}_{\hat{X}_0}[\mathbb{E}_{\hat{X}_{(\cdot)}}[f(\hat{X}_0, \hat{X}_{(\cdot)})]], \qquad (5)$$

f is the notation for n-th the moment;
f is depends on $\hat{X}_0$ and $\hat{X}_{(\cdot)}$, $t \in [0, \Delta t]$;
$\mathbb{E}_{\hat{X}_0}$ is the expectation on random variable $\hat{X}_0$;
$\mathbb{E}_{\hat{X}_{(\cdot)}}$ is the expectation on the process $\hat{X}_{(\cdot)}$ with some fixed initial condition.

Calculation of moments: basic approximation

The equation (4) exactly can not be solved for arbitrary α and β. Used approximate formula [3]:

$$\mathbb{E}_{\hat{X}_{(\cdot)}}[f(\hat{X}_0, \hat{X}_{(\cdot)})] \approx J(f, Y) =$$

$$= \frac{1}{2}\sum_{j=1}^{2} A_j \int\limits_0^1 \int\limits_0^1 \int\limits_{-1}^1 f(\hat{X}_0, Y_j(\cdot, u_1, u_2, v)) du_1 du_2 dv, \qquad (6)$$

Calculation of moments: basic approximation

where

$$Y_t = X_0 + \alpha\Bigg(X_0 + \alpha\left(X_0 + \beta(X_0, |v|)\mathrm{sign}(v)1_{(|v|,1]}(u_2), u_2\right) a_{j2}1_{(u_2,1]}(u_1)$$

$$+\beta\left(X_0 + \alpha(X_0, u_2)a_{j2}1_{(u_2,1]}(|v|), |v|\right) \mathrm{sign}(v)1_{(|v|,1]}(u_1), u_1\Bigg)\rho_{j,1}(t, u_1)$$

$$+\alpha(X_0 + \alpha\left(X_0 + \beta(X_0, |v|)\mathrm{sign}(v)1_{(|v|,1]}(u_1), u_1\right) a_{j1}1_{(u_1,1]}(u_2)$$

$$+\beta\left(X_0 + \alpha(X_0, u_1)a_{j1}1_{(u_1,1]}(|v|), |v|\right) \mathrm{sign}(v)1_{(|v|,1]}(u_2), u_2\Bigg)\rho_{j,2}(t, u_2)$$

$$+\beta\Bigg(X_0 + \alpha\left(X_0 + \alpha(X_0, u_2)a_{j2}1_{(u_2,1]}(u_1), u_1\right) a_{j1}1_{(u_1,1]}(|v|)$$

$$+\alpha\left(X_0 + \alpha(X_0, u_1)a_{j1}1_{(u_1,1]}(u_2), u_2\right) a_{j2}1_{(u_2,1]}(|v|), |v|\Bigg)\rho(t, v),$$

Calculation of moments: basic approximation

$$\rho_{jk}(s, u_k) = a_{jk}1_{[u_k,1]}(s), k = 1, 2, \quad \rho(s, v) = \mathrm{sign}(v)1_{[|v|,1]}(s)$$

$$A_1 + A_2 = 1,$$

$$a_{11} = \frac{1}{2}\left(1 - \sqrt{-\frac{A_2}{A_1}}\right), \quad a_{12} = \frac{1}{2}\left(1 + \sqrt{-\frac{A_2}{A_1}}\right),$$

$$a_{21} = \frac{1}{2}\left(1 - \sqrt{-\frac{A_1}{A_2}}\right), \quad a_{12} = \frac{1}{2}\left(1 + \sqrt{-\frac{A_1}{A_2}}\right),$$

After application of the (6) we can calculate $\mathbb{E}_{\hat{X}_0}$. The probability distribution function (PDF) of $\hat{X}_0$ can be restored by its moments (see, e.g. [4])

Accuracy of the approximation

According to Theorem 1, [3], the accuracy of the calculated value is $\mathcal{O}(\Delta t^{3/2})$. This is insufficient if we want to calculate mathematical expectations through several steps on Δt.
In our case functional f consists of various products of integrals which is already depends on Δt. Therefore common error of application of the formula (6) to the members of f gave us better accuracy.
Example: $\mathbb{E}_{\hat{X}_{(\cdot)}}[\hat{X}_{\Delta t} - \hat{X}_0]$

$$\mathbb{E}_{\hat{X}_{(\cdot)}}[\hat{X}_{\Delta t} - \hat{X}_0] = \mathbb{E}_{\hat{X}_{(\cdot)}}\left[\int_0^{\Delta t} \alpha(\hat{X}_s)ds\right] = \int_0^{\Delta t} \mathbb{E}_{\hat{X}_{(\cdot)}}\left[\alpha(\hat{X}_s)\right] ds =$$

$$\int_0^{\Delta t} J[\alpha, Y] + \mathcal{O}(\Delta t^{3/2})ds = \int_0^{\Delta t} J[\alpha, Y]ds + \mathcal{O}(\Delta t^{5/2})$$

So, PDF of the process can be calculated by moments.
Due to demonstration of the application of the proposed method the normal distribution will be used as an approximation of PDF of $\hat{X}_t$, because for each Δt we have some functional which depends on $W_{\Delta t}$.
So we need to calculate first and second moments of $\hat{X}_{\Delta t}$:

$$\mathbb{E}\hat{X}_{\Delta t}, \quad \mathbb{D}\hat{X}_{\Delta t} = \mathbb{E}[(\hat{X}_{\Delta t} - \mathbb{E}\hat{X}_{\Delta t})^2]$$

Calculation example

Using properties of stochastic Ito's integral $\mathbb{E}\hat{X}_{\Delta t}$ and $\mathbb{D}\hat{X}_{\Delta t}$ can be rewritten as

$$\mathbb{E}\hat{X}_{\Delta t} = \mathbb{E}\hat{X}_0 + \int\limits_0^{\Delta t} \mathbb{E}\alpha(\hat{X}_s)ds, \quad \mathbb{D}\hat{X}_{\Delta t} = \mathbb{D}\hat{X}_0 + B_1 + B_2 + B_3,$$

$$B_1 = \int\limits_0^{\Delta t}\int\limits_0^{\Delta t} \mathbb{E}[\alpha(\hat{X}_{s_1})\alpha(\hat{X}_{s_2})]ds_1 ds_2 - \left(\int\limits_0^{\Delta t} \mathbb{E}\alpha(\hat{X}_s)ds\right)^2,$$

$$B_2 = \int\limits_0^{\Delta t} \mathbb{E}[\beta^2(\hat{X}_s)]ds,$$

$$B_3 = \mathbb{E}\left[\hat{X}_0 \int\limits_0^{\Delta t} \alpha(\hat{X}_s)ds\right] - \mathbb{E}\hat{X}_0 \int\limits_0^{\Delta t} \mathbb{E}[\alpha(\hat{X}_s)]ds$$

Calculation example

Approximate values of $\mathbb{E}\hat{X}_{\Delta t}$ and $\mathbb{D}\hat{X}_{\Delta t}$ can be calculated by consecutive using of (6) and PDF of $\hat{X}_0$.

Now, the iterative scheme of calculation can be used:

- At the first step $\hat{X}_0 \equiv X_0 \in \mathbb{R}$.
- At the next step $\mathbb{E}\hat{X}_{\Delta t}$ and $\mathbb{D}\hat{X}_{\Delta t}$ will be used for restoring of PDF of $\hat{X}_{\Delta t}$, which will be used at the following step as $\hat{X}_0$.

Let us suppose, that $\Delta t = t/N$, where N is number of steps, and an error of PDF approximation is equal or better than $\mathcal{O}(\Delta t^{5/2})$ then the error of proposed iterative scheme is $\mathcal{O}\left(\frac{t^{5/2}}{N^{3/2}}\right)$

Examples

Assumptions

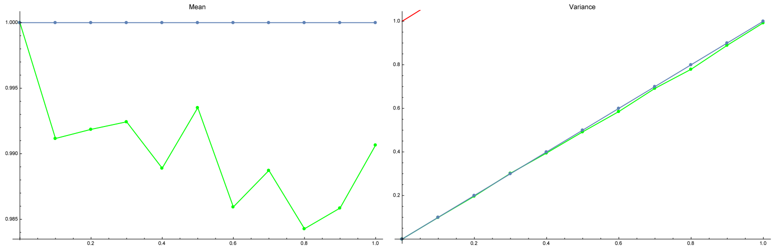
- $t = 1$, $N = 10$ so $\Delta t = 0.1$

At the pictures

- blue curve is the values of the mean and the variance calculated by proposed method
- green curve is the values calculated by Monte Carlo methods with the Milstein's scheme, the number of trajectories is 10000
- red (if it possible) is the exact value of mean or variance

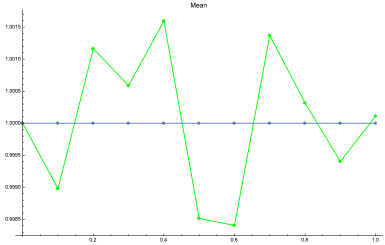
Example: Brownian motion

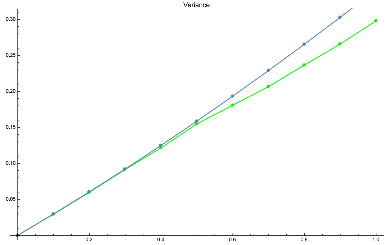
$$\alpha(x) = 0 \quad \beta(x) = 1$$



Example: Nonlinear SDE without drift

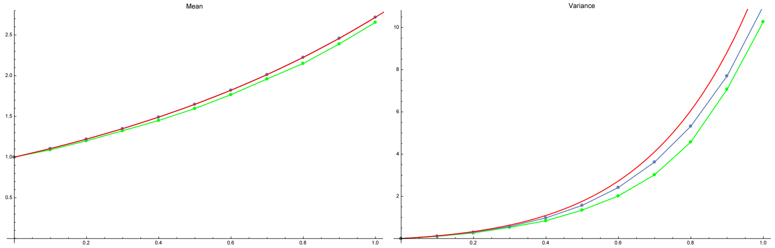
$$\alpha(x) = 0 \quad \beta(x) = cosx$$





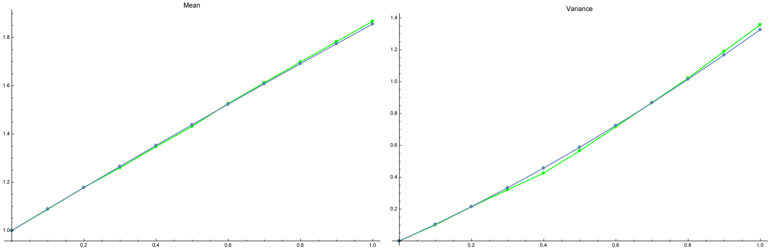
Example: Linear SDE

$$\alpha(x) = x \quad \beta(x) = x$$



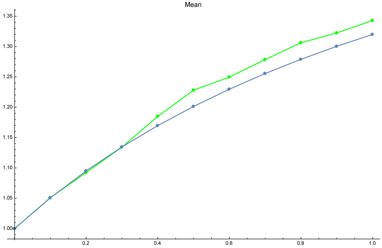
Example: Cox-Ingersoll-Ross model

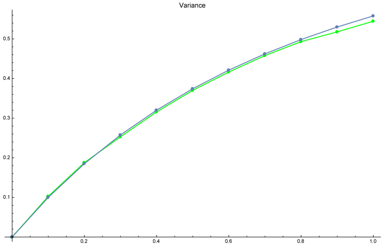
$$\alpha(x) = 1 - 0.1x \quad \beta(x) = \sqrt{x}$$



Example: Nonlinear drift with white noise

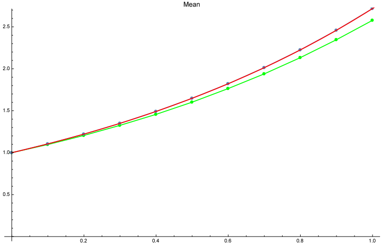
$$\alpha(x) = cosx \quad \beta(x) = 1$$

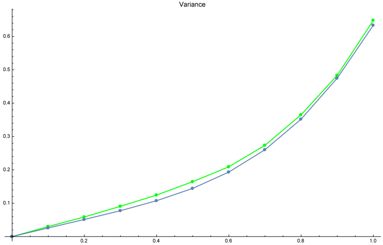




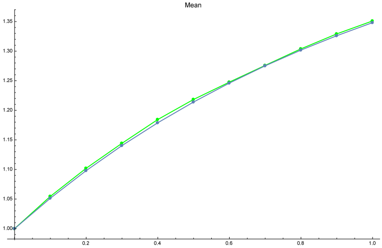
Example: Nonlinear SDE with linear drift

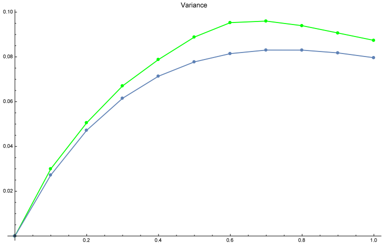
$$\alpha(x) = x \quad \beta(x) = cosx$$





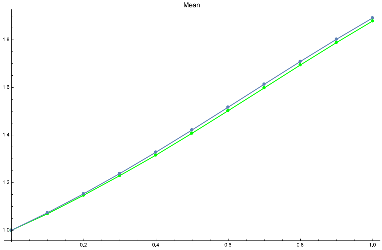
$$\alpha(x) = cosx \quad \beta(x) = cosx$$

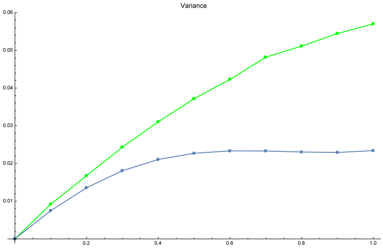




Example: Another nonlinear SDE

$$\alpha(x) = sin^2 x \quad \beta(x) = cos^2 x$$





Refferences

- [1] B. Øksendal. Stochastic Differential Equations: An Introduction with Applications. Springer, 2003.
- [2] D. Applebaum. Levy processes and stochastic calculus. Cambridge University Press, 2009.
- [3] Anatoly Zherelo, Approximate Calculation of Mathematical Expectations on Process with a Drift // Nonlinear Phenomena in Complex Systems, vol.18, no.2, pp.207–214.
- [4] Juraj Tekel, Juraj Tekel, Constructing and Estimating Probability Distributions from Moments // Proceedings of SPIE, vol. 8391, May 2012.