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A Finding of Switching Instants for Dynamical Polysystems by Applying Continued Fractions

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Control systems with a finite number of control settings are considered. It is assumed that each polysystem operates in continuous time and control switchings occur at some certain discrete time instants. The control goal is to transfer the polysystem from an initial state to a final state. Controllability of systems switched in discrete time is studied. Controls are constructed by using the theory of generalized continued fractions and the congruence theory.

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1. Introduction

For control of system in discrete-time, it is necessary to determine the switching instants at which control actions are occurred. Besides, some additional restrictions must be taken into account. For example, the control switching instants should be integer provided that the control accuracy is ensured. The problem of finding integer switching instants can be successfully solved for a polysystem with some stochastic properties. However, their presence is usually difficult to verify. Without the assumption of stochastic properties, in some cases, the problem of finding integer switching instants can be reduced to the problem of solving Diophantine equations or inequalities. Note, that it is usually possible to find solutions of these equations only when the number of unknown switching instants is small, i.e. for cases of small dimension of a state space, [1–4]. For cases of significant dimensionality of a state space, these methods lead to a loss of control accuracy. For these cases, the development of specific methods for finding of integer switching instants is required.

In this paper, we propose a method that allows us to solve the problem of finding integer switching instants for some cases of significant dimensionality of the state space. This method is based on the theory of multidimensional continued fractions.

2. Basic definitions and mathematical formulation of the problem

Let $\mathbb{X} = \{x\}$ be an m-dimensional state space, $\mathbb{R} = \{t\}$ be a one-dimensional time space. A set of maps

$$F_i^t : \mathbb{X} \to \mathbb{X}, \quad t \in \mathbb{R}, i = 1, \dots, l, l \in \mathbb{N} \quad (2.1)$$

with a semigroup property is called a dynamical polysystem.

Let $t_{j-1} \leq t_j$, $j = 1, \dots, l$, $l \geq m$ be switching instants. On time interval $(t_{j-1}, t_j]$, the dynamical process is given by one of the elements of the set $\{F_i^t\}$.

Let $x_0 = x(t_0)$ be a initial state. Let $\tau_j := t_j - t_{j-1} \geq 0$, $j = 1, \dots, l$. The state of the polysystem at the last instant of time is as follows

$$(F_{i_l}^{\tau_l} \circ F_{i_{l-1}}^{\tau_{l-1}} \circ \dots \circ F_{i_1}^{\tau_1})(x_0) := F(\tau, x_0), \quad (2.2)$$

where $\tau = (\tau_1, \dots, \tau_l) \in \mathbb{R}^l$. The set $\mathbb{R}^l$ is said to be an l-dimensional time space.

The polysystem is ϵ-controllable from the state x_0 to the state x_* if for every $\epsilon > 0$ there

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exists $\tau = (\tau_1, \dots, \tau_l) \in \mathbb{R}^l_+$ depending on ϵ, i.e. $\tau = \tau(\epsilon)$, such that

$$|F^\tau(x_0) - x_*| \leq \epsilon. \tag{2.3}$$

The full control time is $|\tau| = |\tau_1| + \cdots + |\tau_l|$. If for some τ the value $\epsilon = 0$ then the polysystem is called precisely controlled.

Let some polysystem be precisely controllable. Thus, $\epsilon = 0$ and $\tau(0) \in \mathbb{R}^l$. Our goal is to show that this polysystem is ϵ-controllable also in discrete time. Thus, for any ϵ the switching instants $\tau(\epsilon) \in \mathbb{R}^l_+$ can be selected as integers.

In the following sections, we consider the mathematical foundations of the necessary constructions. Below, we consider some classes of polysystems (see sec. 7) where we indicate a method of finding of the switching instants $\tau(\epsilon)$.

3. Ordinary continued fractions

An ordinary continued fraction can be written in the form a_0+

$$+(a_1 + \cdots + (a_{n-1} + (a_n + \dots)^{-1})^{-1} \dots)^{-1} \tag{3.1}$$

where $a_n \in \mathbb{N}$, $n = 0, 1, \dots$. Short records are $(a_0, a_1, \dots, a_n, \dots)$ or $a_0 + \frac{1}{a_1 + \dots}$. For a finite continued fraction, the term $(a_n + \dots)$ is replaced by the term (a_n). A finite continued fraction can be written as an ordinary fraction in the form $r_n = \frac{p_n}{q_n}$ where the numerator and denominator are respectively $p_n = \mathcal{N}(r_n)$ and $q_n = \mathcal{D}(r_n)$. From these numbers, it is possible to generate the sequence $(r_0, r_1, \dots, r_n, \dots)$ which is called a sequence of convergents of a continued fraction.

By direct calculations, using the method of mathematical induction, it can be shown that p_n and q_n are calculated recursively, i.e. they satisfy the equations

$$q_{-1} = 0,\, q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2}, \tag{3.2}$$

$$p_{-1} = 1,\, p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-2}. \tag{3.3}$$

4. Spaces which elements can are represented by continued fractions

Consider some set in which there are two algebraic operations, namely addition and multiplication. With respect to addition, the set is an abelian group, and with respect to multiplication, it is an arbitrary group. In addition, we assume that this set also is a Euclidean space $\mathbb{R}^m$ of dimension m over the field $\mathbb{R}$ of reals. This set together with the introduced structures is denoted by $\mathbb{T}$.

For example, as the set $\mathbb{T}$, the sets $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ can be considered, i.e. respectively, real numbers, complex numbers, Hamilton numbers (quaternions), Cayley numbers (octanions), as well as the sets of square non-degenerate matrices of arbitrary order.

The following notation is used below. Let $\mathfrak{G}$ be a lattice of integer elements $\mathbb{R}^m$, i.e. $\mathfrak{G} := \mathbb{Z}^m$ and $\mathbb{T}^0 := \mathbb{Q}^m$ where $\mathbb{Q}$ is the field of rational numbers. Let $\mathbb{T}^* := \mathbb{T} \backslash \{0\}$, $\mathbb{T}' := \mathbb{T} \backslash \mathbb{T}^0$. Elements of $\mathbb{T}^*$ can are represented by continued fractions.

5. Generalized continued fractions

In this section, we consider generalized continued fractions, which are elements of the space $\mathbb{T}$ introduced in the previous section. Thus, generalized continued fractions are multidimensional. We define a generalized continued fraction through an iterative sequence. An iterative sequence $\{z_n\}$ is formed more loosely (less restrictive) than an iterative sequence used for determining of classical continued fractions.

5.1. Iteration sequences

Let $f : \mathbb{T}^* \mapsto \mathfrak{G}$ be some function called a choice function. Using the map f, we associate to each $z \in T$ two sequences which are determined inductively. Namely, the sequence $\{z_n\}$ and the sequence $\{a_n\}$ are given as follows. First, we define $z_0 = z$ and $a_0 = f(z_0)$. Next, let $z_0, ..., z_n$

and $a_0, ..., a_n$ be defined for $n \geq 0$ where $a_k = f(z_k), k = 0, 1, \ldots, n$. If $f(z_n) = z_n$ we terminate these sequences and, otherwise, we define the following values

$$z_{n+1} = (z_n - f(z_n))^{-1}, \quad a_{n+1} = f(z_{n+1}). \tag{5.1}$$

Notice, the first of condition (5.1) can be rewritten in the form

$$a_n = z_n - z_{n+1}^{-1}, \quad n \geq 0. \tag{5.2}$$

A sequence $\{z_n\}_{n=0}^{\infty}$ is called an iteration sequence for a number z if $z_0 = z$, $z_0 - z_1^{-1} \in \mathfrak{G}$, and for $n = 1, 2, \ldots$

$$|z_n| \geq 1, \quad z_n - z_{n+1}^{-1} \in \mathfrak{G} \setminus \{0\}. \tag{5.3}$$

A sequence $\{a_n\}_{n=0}^{\infty}$ is called a sequence of partial quotients.

Further, we assume that $z \in \mathbb{T}'$. An iteration sequence is said to be degenerate if there exists n_0 such that $|z_n| = 1$ for all $n \geq n_0$. It is said to be nondegenerate if it is not degenerate.

We assume that the multivalued choice function f can provide that the sequence z_n, $n = 1, 2, \ldots$ is nondegenerate.

For the initial element z and for the algorithm defined by the function f, the sequence $(a_0, a_1, \ldots, a_n, \ldots)$ gives the continued fraction, which is defined by the expression in the form (3.1). The sequence $(a_0, a_1, \ldots, a_n, \ldots)$ is the short form of continuous fraction (3.1).

If there is a finite sequence $(a_0, a_1, \ldots, a_n)$ then it corresponds to a continued fraction r_n where the term $(a_n + \ldots)$ in Eq. (3.1) is replaced by the term (a_n). A continued fraction r_n is a "rational element", i.e. $r_n \in \mathbb{T}^0 = \mathbb{Q}^m$, which is a covergent of an infinity continued fraction. Similarly to classical continued fractions, a covergent fraction r_n can be represent in the form $q_n^{-1} p_n \in \mathbb{T}^0$ where p_n and q_n, respectively, are called a numerator and a denominator of the fraction r_n.

5.2. An auxiliary statements

By calculations, it can be shown that the quantities p_n and q_n satisfy equations of the form (3.3) – (3.2). Note that the commutativity of multiplication is not assumed, i.e. when multiplying, the factors are not rearranged. The following lemma holds which is an analogue of the proposition 3.3 in [5].

Lemma 1 *Assume $z \in \mathbb{T}'$. Let $\{z_n\}$ be an iteration sequence for z, $\{a_n\}$ be the corresponding sequence of partial quotients, and $\{r_n\}$ be the sequence of convergents of the continued fraction. Then we have that for all $n \geq 0$*

$$q_n z - p_n = (-1)^n (z_1 \cdots z_{n+1})^{-1}, \tag{5.4}$$

$$z = (1 + q_n^{-1} z_{n+1}^{-1} q_{n-1})^{-1} (q_n^{-1} p_n) \times (1 + p_n^{-1} z_{n+1}^{-1} p_{n-1}). \tag{5.5}$$

Proof of Lemma 1.

1) We proof Eq. (5.4) by induction. Since $p_0 = a_0, q_0 = 1$, and $z - a_0 = z_1^{-1}$ then the statement holds for $n = 0$. Now for $n \geq 1$, by induction, suppose that the assertion holds for $0, 1, ..., n-1$. Then we have

$$\begin{aligned} q_n z - p_n = (a_n q_{n-1} + q_{n-2}) z - (a_n p_{n-1} + p_{n-2}) = \\ a_n (q_{n-1} z - p_{n-1}) + (q_{n-2} z - p_{n-2}) = \\ a_n (-1)^{n-1} (z_1 \cdots z_n)^{-1} + (-1)^{n-2} (z_1 \cdots z_{n-1})^{-1} = \\ -a_n (-1)^n (z_1 \cdots z_n)^{-1} + (-1)^n z_n (z_1 \cdots z_{n-1} z_n)^{-1} = \\ (-1)^n (-a_n + z_n)(z_1 \cdots z_n)^{-1} \overset{(5.2)}{=} \\ (-1)^n z_{n+1}^{-1} (z_1 \cdots z_n)^{-1} = (-1)^n (z_1 \cdots z_{n+1})^{-1}. \end{aligned} \tag{5.6}$$

2) We proof Eq. (5.5). From relation (5.4) it follows that

$$q_n z - p_n = -z_{n+1}^{-1} (q_{n-1} z - p_{n-1}),$$

whence it follows that

$$(q_n + z_{n+1}^{-1} q_{n-1}) z = p_n + z_{n+1}^{-1} p_{n-1}.$$

Solving this equation with respect to z, we obtain the following equivalent relations

$$z = (q_n + z_{n+1}^{-1} q_{n-1})^{-1}(p_n + z_{n+1}^{-1} p_{n-1}). \quad (5.7)$$

Relation (5.7) can be rewritten in the form

$$z = (1 + q_n^{-1} z_{n+1}^{-1} q_{n-1})^{-1} \times (q_n^{-1} p_n)(1 + p_n^{-1} z_{n+1}^{-1} p_{n-1}). \quad (5.8)$$

Lemma 1 is proved.

The following statement on sufficient conditions for convergence holds.

Proposition 1 *Let the conditions of lemma 1 be satisfied. Put* $|z_n| = 1 + \xi_n$, $n = 1, 2, \dots$ *where* $\xi_n > 0$. *Assume that* $\sum_{n=1}^{\infty} \xi_n = \infty$. *Then condition* $\lim_{n\to\infty} |q_n z - p_n| \to 0$ *is satisfied.*

PROOF of Proposition 1.

From eq. (5.4), $|q_n z - p_n| = \frac{1}{|z_1|\cdots|z_{n+1}|}$. Consider the following cases.

1. Assume that

$$\bar{\xi} := \limsup_{n\to\infty} \xi_n > 0. \quad (5.9)$$

This means that there is a number $\xi_* > 0$ and there is a subsequence of the sequence ξ_n such that all members of this subsequence are greater than ξ_*. Suppose that for each n among the elements $\xi_1, \dots, \xi_{n+1}$ there are k_n elements which are greater than ξ_*. Obviously, $k_n \to \infty$ as $n \to \infty$. Since $|z_1| \geq 1, \dots, |z_{n+1}| \geq 1$ we get that $|z_1| \cdots |z_{n+1}| \geq (1+\xi_*)^{k_n}$. Therefore, for $n \to \infty$, we have that

$$|q_n z - p_n| = \frac{1}{|z_1| \cdots |z_{n+1}|} \leq \frac{1}{(1+\xi_*)^{k_n}} \to 0.$$

2. Assume that $\limsup \xi_n = 0$. Therefore, for any ξ_*, there is only a finite number of members of the iterative sequence for which $\xi_n > \xi_*$. Let ξ_* be the root of the equation $\ln \xi_* = \xi_*$. Easy to get what $\xi_* \approx 1.3$. Obviously, there exists a natural n_0 such that for $n \geq n_0$ the inequalities $0 < \xi_n \leq \xi_*$ are satisfied. For simplicity, we assume that $n_0 = 1$. Then

$$|z_1| \cdots |z_{n+1}| = e^{\ln(1+\xi_1)+\cdots+\ln(1+\xi_{n+1})} > e^{\xi_1+\cdots+\xi_{n+1}}.$$

Since $\xi_1 + \cdots + \xi_{n+1} \to +\infty$ as $n \to \infty$ then $|q_n z - p_n| \to 0$ as $n \to \infty$.

Proposition 1 is proved.

5.3. Volume forms defined on sequences of convergent fractions

Let $r_{n-1} = q_{n-1}^{-1} p_{n-1}$ and $r_n = q_n^{-1} p_n$ be two neighboring convergent fractions where $n = 0, 1, \dots$. Consider the determinant

$$v_{n,n-1} := \begin{vmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{vmatrix}. \quad (5.10)$$

Let the set $\mathbb{T}$ be a field. Then

$$v_{n,n-1} = q_n p_{n-1} - q_{n-1} p_n. \quad (5.11)$$

The value $v_{n,n-1}$ is interpreted as $2d$-volume of parallelogram which spans the vectors $\vec{r}_{n-1} = \mathrm{col}(q_{n-1}, p_{n-1})$ and $\vec{r}_n = \mathrm{col}(q_n, p_n)$.

From formulas (3.2), (3.3), it follows that $v_{n,n-1} = (-1)^n$

Let $z \in \mathbb{R}$. Then the following inequalities are true, [6].

$$|r_n - z| < |r_{n-1} - z| < |r_n - r_{n-1}| < 2|r_{n-1} - z|. \quad (5.12)$$

Since $|v_{n,n-1}| = 1$, from (5.12), there are following upper and lower bounds as follows

$$|r_{n-1} - z| < \frac{1}{q_n q_{n-1}} < \frac{1}{q_{n-1}^2}, \quad (5.13)$$

$$\frac{1}{2q_n^2} < \frac{1}{2q_{n-1}^2} < |r_{n-1} - z|. \quad (5.14)$$

Now, consider generalized continued fractions. We do not assume that the elements of this determinant are commutative. Therefore, we assume that the value of this determinant is calculated by Schur's formulas, see [7], p. 46. Thus, by definition, we assume that for

$n = 0, 1, 2, \ldots$

$$v_{n,n-1} = q_n p_{n-1} - q_n q_{n-1} q_n^{-1} p_n. \qquad (5.15)$$

The value $v_{n,n-1}$ is interpreted as $2d$- "symbolic volume" of parallelogram which spans the vectors $\vec{r}_{n-1} = \mathrm{col}(q_{n-1}, p_{n-1})$ and $\vec{r}_n = \mathrm{col}(q_n, p_n)$. The value $v_{n,n-1}$ given by Eq. (5.15) can also be rewritten as

$$v_{n,n-1} = q_n q_{n-1} q_{n-1}^{-1} p_{n-1} - q_n q_{n-1} q_n^{-1} p_n = q_n q_{n-1}(r_{n-1} - r_n). \qquad (5.16)$$

By calculations it can be shown that

$$v_{0,-1} = 1,\ v_{1,0} = -1,\ v_{2,1} = 1 + \frac{q_2}{|q_2|^2}[a_2, a_1], \ldots,$$

where $[a_2, a_1] = a_2 a_1 - a_1 a_2$. The value $\frac{1}{|q_2|^2} q_2 [a_2, a_1]$ is the magnitude of distortion of the volume. If $[a_2, a_1] = 0$ then $v_{2,1} = 1$, i.e., there is no distortion of volume.

6. The main result on the convergence of multidimensional continued fractions

Proposition 1 establishes only that the continued fraction converges to the element z, but does not give an estimate of the rate of convergence. Using the characteristic $v_{n,n-1}$ given in sec. 5 5.3 we get the estimate of the rate of convergence of a continued fraction in the following theorem.

Theorem 1 *Suppose that the conditions of lemma 1 are satisfied and, in addition, in formula (5.5) for sufficiently large $n \geq 0$*

$$|q_n^{-1} z_{n+1}^{-1} q_n| < 1. \qquad (6.1)$$

Then the following asymptotic equality holds for $n \geq 0$

$$\left| z - q_n^{-1} p_n \right| = \frac{|v_{n,n-1}|}{|q_n|^2 |z_{n+1}|} + o\left(\frac{1}{|q_n|^2 |z_{n+1}|} \right). \qquad (6.2)$$

Proof of Theorem 1. Let $r_n = q_n^{-1} p_n$. By assumption, the quantity $q_n^{-1} z_{n+1}^{-1} q_{n-1}$ is small in equality (5.5). Then we get the following approximate equality

$$z = (1 - q_n^{-1} z_{n+1}^{-1} q_{n-1}) r_n (1 + p_n^{-1} z_{n+1}^{-1} p_{n-1}),$$

which we transform discarding the terms of the second order of smallness as follows

$$z = r_n - q_n^{-1} z_{n+1}^{-1} q_{n-1} r_n + r_n p_n^{-1} z_{n+1}^{-1} p_{n-1}.$$

Consider the discrepancy $z - r_n$ where $r_n = q_n^{-1} p_n$. Then

$$\begin{aligned} z - r_n &= q_n^{-1} z_{n+1}^{-1} p_{n-1} - q_n^{-1} z_{n+1}^{-1} q_{n-1} q_n^{-1} p_n = \\ &= q_n^{-1} z_{n+1}^{-1} (p_{n-1} - q_{n-1} q_n^{-1} p_n) = \\ &= q_n^{-1} z_{n+1}^{-1} q_n^{-1} (q_n p_{n-1} - q_n q_{n-1} q_n^{-1} p_n). \end{aligned}$$

Taking into account condition (5.15), we get the estimate of the norm of the residual up to terms of the second order of smallness as follows

$$|z - r_n| = \frac{|q_n p_{n-1} - q_n q_{n-1} q_n^{-1} p_n|}{|q_n|^2 |z_{n+1}|} = \frac{|v_{n-1,n}|}{|q_n|^2 |z_{n+1}|}, \qquad (6.3)$$

The theorem 1 is proved.

Comment. The assumption (6.1) means that the value $\overline{\xi}$ in formula (5.9) sufficiently large and valueses $|q_n| \cdot |q_{n-1}|$ are limited, i.e. continued fraction converges quickly enough.

Corollary. From formula (5.16), it follows that

$$|v_{n-1,n}| = |q_n q_{n-1}(r_{n-1} - r_n)| \leq \qquad (6.4)$$
$$|q_n q_{n-1}|(|r_{n-1} - z| + |r_n - z)|). \qquad (6.5)$$

Thus, from formulas (6.3) and (6.5), the following asymptotic inequalities hold

$$\frac{|v_{n-1,n}|}{2|q_n q_{n-1}|} \lesssim |z - r_n| \lesssim \frac{|v_{n-1,n}|}{|q_n|^2},$$

which are an analogues of formulas (5.13) and (5.14).

7. The application of continued fractions and the theory of congruences for control

In order to find Diophantine solutions $\tau = (\tau_1, \dots, \tau_l)$ of inequality (2.3) we have to solve an auxiliary problem of approximation of a set of real numbers by a set of rational numbers with the same denominator. These problems can be solved by using continued fractions and congruence theory.

Consider relation (2.3) for $\epsilon = 0$, i.e. the equation $F^\tau(x_0) = x_*$ which can also be written as $\tilde{F}(x_0, x_*; \tau) = 0$. We shall study polysystem (2.1) such that $\tilde{F}(x_0, , x_*; \tau)$ is an asymptotic polynomial in τ for sufficiently large τ. In particular, $\tilde{F}(x_0, x_*; \tau)$ is a linear function in τ, i.e. it is a superposition of stationary shifts of the form $x \to x + at$ where $a \in \mathbb{R}^m$ is a constant vector, t is time.

These and more complex dependencies $\tilde{F}$ arise for some classes of linear polysystems. For example, if $x' \to e^{at}x'$ is an exponential map then $x \to x + at$ is a shift where $x = \ln x'$. If, among the elements of some linear polysystem (2.1), there are matrix exponents with non-commuting matrices then it can lead to asymptotically linear or asymptotically polynomial shifts in logarithmic coordinates.

Further, we mainly consider the case, when the equation $\tilde{F}(x_0, x_*; \tau) = 0$ can be represented as follows

$$\psi(x_0, x_*) + \Psi(x_0, x_*)\tau + T_{x_0,x_*}(\tau) = 0, \quad (7.1)$$

where $T_{x_0,x_*}(\tau) \to 0$ as $\tau \to \infty$.

Assume that $\operatorname{rank} \Psi(x_0, x_*) = m$. Let $\tau' = (\tau_1, \dots, \tau_m)$, $\tau'' = (\tau_{m+1}, \dots, \tau_l)$, and $\det[\frac{\partial \Psi}{\partial \tau'}] \neq 0$. Then equation (7.1) is solvable with respect to τ' in the form

$$\tau' = \varphi^0(x_0, x_*) + \Phi(x_0, x_*)\tau'' + T_{x_0,x_*}(\tau''), (7.2)$$

where $\Phi = (\varphi^1, \dots, \varphi^k)$ and $T_{x_0,x_*}(\tau'') \to 0$ as $\tau'' \to \infty$.

Next, we consider the linear part of equality (7.2) for sufficiently large $|\tau''|$. Let the x_0, x_* be fixed. Further, we omit the x_0, x_* in following formulas. Rename τ_{m+j} by s_j, $j = 1, \dots, k$ where $m + k = l$. Then $\tau'' = \operatorname{row}(s_1, \dots, s_k)$. From Eq. (7.2), it follows that

$$\tau' = \varphi^0 + \varphi^1 s_1 + \dots + \varphi^k s_k. \quad (7.3)$$

Next, we approximate the real vectors φ^j by the rational vectors r^j respectively depending on n with a common denominator, i.e., for $j = 0, 1, \dots k$,

$$r^j := \frac{\hat{\rho}^j}{q} \in \mathbb{Q}^m, \quad \hat{\rho}^j \in \mathbb{Z}^m, \quad q \in \mathbb{N}. \quad (7.4)$$

For this, we use a continued fractions of the form (3.1) for some value $n \in \mathbb{N}$ which provides sufficient accuracy of approximation $\varphi^j \approx r^j$. Thus, the rational vector

$$\varrho = \frac{1}{q}\left(\hat{\rho}^0 + \hat{\rho}^1 s_1 + \dots + \hat{\rho}^k s_k\right) \in \mathbb{Q}^m \quad (7.5)$$

approximates the real vector τ' given by Eq. (7.3), i.e. $\tau' \approx \varrho$.

Further, we suppose that the values of $s_1, \dots, s_k \in \mathbb{N}$ can be chosen such that the components of the vector r are positive integers. For this, the vector $\hat{\rho}^0 + \hat{\rho}^1 s_1 + \dots + \hat{\rho}^k s_k$ should be divided component-wise by the integer q. In other words, the positive integers $s_1, \dots, s_k$ must satisfy the vector equation for the congruence modulo q of the form

$$\hat{\rho}^0 + \hat{\rho}^1 s_1 + \dots + \hat{\rho}^k s_k \equiv 0 \mod q. \quad (7.6)$$

Conditions are known (see [6]) under which equation (7.6) can be solved, i.e., there are constants $c_1, \dots, c_k$ such that

$$s_1 \equiv c_1 \mod (q), \dots, s_k \equiv c_k \mod (q). (7.7)$$

As a result, we obtain an integer approximation, i.e. $\tau = (\tau', \tau'') \approx (\overline{\varrho}, \overline{s}) \in \mathbb{N}^l$ where $\overline{s} = (\overline{s}_1, \dots, \overline{s}_k) \in \mathbb{N}^k$ is the smallest element of residue class (7.7), i.e. the value $|\overline{s}|$ is minimal.

Then value

$$\overline{\rho} = \frac{\hat{\rho}^0 + \hat{\rho}^1 \overline{s}_1 + \cdots + \hat{\rho}^k \overline{s}_k}{q} \in \mathbb{N}^m. \qquad (7.8)$$

It is especially simple to find a solution of vector equation (7.6) for the case $m = k$. In the following example, we consider the case $m = k = 1$, i.e. the scalar equation

$$\rho^0 + \rho^1 s \equiv 0 \mod q. \qquad (7.9)$$

The ordinary rational fraction $\frac{\rho^1}{q}$ can be represented as a continued fraction. Suppose that for some positive integer ν depending on a given accuracy of calculations, there is a finite sequence of convergent fractions $(r_0^1, r_1^1, \dots, r_{\nu-1}^1, r_\nu^1)$. Here $r_\nu^1 := \frac{\rho_\nu^1}{q_\nu^1} = \frac{\rho^1}{q}$ is the latest convergent fraction, $r_{\nu-1}^1 := \frac{\rho_{\nu-1}^1}{q_{\nu-1}^1}$ is the penultimate convergent fraction, values of the remaining convergents are not significant. Since (see [6]),

$$q_\nu^1 \rho_{\nu-1}^1 - q_{\nu-1}^1 \rho_\nu^1 = (-1)^\nu, \quad \nu = 0, 1, \dots,$$

there is the following “chain of congruences” modulo $\mod q$:

$$q_{\nu-1}^1 \rho_\nu^1 \equiv (-1)^{\nu-1} \Rightarrow (-1)^{\nu-1} q_{\nu-1}^1 \rho_\nu^1 \equiv 1 \Rightarrow$$
$$(-1)^{\nu-1} q_{\nu-1}^1 \rho_\nu^1 s \equiv s \overset{(7.9)}{\Rightarrow} (-1)^{\nu-1} q_{\nu-1}^1 (-\rho^0) \equiv s$$
$$\Rightarrow (-1)^\nu q_{\nu-1}^1 \rho^0 \equiv s.$$

Let $\tilde{q}^1 := q_{\nu-1}$. Note, $\tilde{q}^1$ depends on q. Thus, the residue class

$$s \equiv (-1)^\nu \rho^0 \tilde{q}^1 =: c \mod q \qquad (7.10)$$

is the solution of original congruence (7.9). The smallest positive number $\overline{s}$ from residue class (7.10) can be easily calculated.

Now, we consider the general case $k = m$ for an arbitrary m. It is easy to show that vector equation (7.6) is equivalent to the following system of equations

$$\Delta_i^0 + \Delta^1 s_i \equiv 0, \quad i = 1, \dots, m, \qquad (7.11)$$

where

$$\Delta^1 = \det(\hat{\rho}^1, \dots, \hat{\rho}^i, \dots \hat{\rho}^k), \qquad (7.12)$$
$$\Delta_i^0 = \det(\hat{\rho}^1, \dots, \hat{\rho}^0, \dots \hat{\rho}^k).$$

Taking into account (7.10), the solution of system (7.11) can be represented as follows

$$s_i \equiv (-1)^\nu \Delta_i^0 \tilde{\Delta}^1 \mod q, \ i = 1, \dots, m, \qquad (7.13)$$

where $\tilde{\Delta}^1$ is the numerator of the penultimate convergent of continued fraction $\frac{\Delta}{q}$. Note, $\tilde{\Delta}^1$ depends on q. Thus, in formula (7.7), the values $c_i = (-1)^\nu \Delta_i^0 \tilde{\Delta}^1 \mod q$ where $i = 1, \dots, m$.

Example. Let the two-dimensional state space $\mathbb{R}^2$ be represented as the complex plane $\mathbb{C}$. Consider the polysystem defined by the mappings of the complex plane $\mathbb{C} \to \mathbb{C}$ as follows

$$z \to e^{(\lambda_\kappa + i\beta_\kappa)t} z, \quad \kappa = 1, 2, 3, 4, \qquad (7.14)$$

where the i is the imaginary unit, the parameter t is real time. The map given by Eq. (7.14) can be easily rewritten in the matrix real form.

Let us assume that the starting point z_0 should be transferred to the final point z_* using the map defined by an equation of the form

$$e^{(\lambda_1 + i\beta_1)\tau_1} e^{(\lambda_2 + i\beta_2)\tau_2} e^{(\lambda_3 + i\beta_3)\tau_3} e^{(\lambda_4 + i\beta_4)\tau_4} z_0 = z_*,$$

which can be rewritten as a system of two equations as follows

$$\lambda_1 \tau_1 + \lambda_2 \tau_2 + \lambda_3 \tau_3 + \lambda_4 \tau_4 = \ln \left| \frac{z_*}{z_0} \right|,$$
$$\beta_1 \tau_1 + \beta_2 \tau_2 + \beta_3 \tau_3 + \beta_4 \tau_4 = \arg \frac{z_*}{z_0}.$$

Suppose that the matrix of this system have rank two. Taking into account formula (7.3) for $m = 2$ and $k = 2$, the values τ_1, τ_2 can be expressed through the parameters $s_1 := \tau_3, s_2 := \tau_4$ as follows

$$\tau_1 = \varphi_1^0 + \varphi_1^1 s_1 + \varphi_1^2 s_2, \qquad (7.15)$$
$$\tau_2 = \varphi_2^0 + \varphi_2^1 s_1 + \varphi_2^2 s_2. \qquad (7.16)$$

The set of real values $(\varphi_1^0, \varphi_1^1, \varphi_1^2; \varphi_2^0, \varphi_2^1, \varphi_2^2)$

can be approximated by the set of convergent fractions $(r_1^0, r_1^1, r_1^2; r_2^0, r_2^1, r_2^2)$ with same denominators q, i.e. $r_i^j = \frac{\rho_i^j}{q}$ where $i = 1, 2$ and $j = 0, 1, 2$. For this purpose, multicomponent continued fractions can be used.

Next, to the system of real equations (7.15), (7.16) is assigned the system of equations for comparisons as follows

$$\rho_1^0 + \rho_1^1 s_1 + \rho_1^2 s_2 \equiv 0, \quad \rho_2^0 + \rho_2^1 s_1 + \rho_2^2 s_2 \equiv 0.$$

The solutions of this system can be written as

$$s_1 \equiv (-1)^\nu \Delta_1^0 \tilde{\Delta}^1 \mod q, \qquad (7.17)$$
$$s_2 \equiv (-1)^\nu \Delta_2^0 \tilde{\Delta}^1 \mod q, \qquad (7.18)$$

where

$$\Delta^1 = \begin{vmatrix} \rho_1^1 & \rho_1^2 \\ \rho_2^1 & \rho_2^2 \end{vmatrix}, \quad \Delta_1^0 = \begin{vmatrix} \rho_1^0 & \rho_1^2 \\ \rho_2^0 & \rho_2^2 \end{vmatrix}, \quad \Delta_2^0 = \begin{vmatrix} \rho_1^1 & \rho_1^0 \\ \rho_2^1 & \rho_2^0 \end{vmatrix},$$

and ν is a number of covergent fractions for decomposing of the fraction $\frac{\Delta^1}{q}$ into a continued fraction. Thus, $\mathrm{col}(\tau_1, \tau_2) \approx (\overline{\varrho}_1, \overline{\varrho}_2)$, $\mathrm{col}(\tau_3, \tau_4) \approx (\overline{s}_1, \overline{s}_2)$ where

$$\overline{\varrho}_1 = \frac{\rho_1^0 + \rho_1^1 \overline{s}_1 + \rho_1^2 \overline{s}_2}{q},$$
$$\overline{\varrho}_2 = \frac{\rho_2^0 + \rho_2^1 \overline{s}_1 + \rho_2^2 \overline{s}_2}{q},$$

$\overline{s}_1$, and $\overline{s}_2$ are the smallest positive elements of the residue classes (7.17) and (7.18) accordingly.

8. Conclusion

In order to solve the stated control problems, some classes of multicomponent continued fractions were introduced. For these classes of continued fractions, convergence conditions were obtained. The application of these fractions to control systems in discrete time has been shown.

The advantage of the multicomponent continued fraction method is that the approximation of a set of real elements $z = (z_1, \ldots, z_n)$ is carried out by a set of rational elements $r = (r_1, \ldots, r_n)$ with the same denominators, i.e. $r_j = \frac{p_j}{q}$. Alternative methods of approximation of real numbers by rational numbers give approximating values with a large number of signs, while ensuring the same accuracy. Note that the proposed method does not require the presence of statistical properties, in contrast to alternative methods of estimation.

There is a program using MatLab which finds integer approximations of the set of real numbers $t = (t_1, \ldots, t_l)$.

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